QCD phase diagram and the equation of state from lattice QCD

Frithjof Karsch
Bielefeld University

UNIVERSITÄT BIELEFELD
Faculty of Physics

Outline

- Exploring the phase diagram of QCD:
  Critical behavior in the limit of vanishing light quark masses
- Calculating the equation of QCD state at non-zero temperature and chemical potentials:
  Taylor expansions & Conserved charge fluctuations
- Constraining the location of the critical point
  resumming Taylor expansions

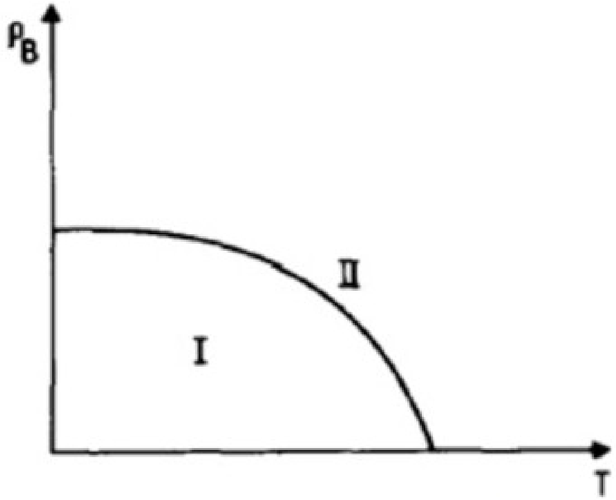
CRC-TR 211 Strong-interaction matter under extreme conditions
DFG Deutsche Forschungsgemeinschaft

Strongly interacting matter in the '70s and early '80s

LGT~1980

Mike Creutz

Phase diagram of QCD



N. Cabibbo, G. Parisi,
Phys. Lett. 59B (1975) 67

HRG~1964

Rolf Hagedorn:
Hadron resonance gas,
ultimate temperature?

- the physics/thermodynamics of strong interaction matter is described by the theory of strong interactions – Quantum Chromo Dynamics (QCD)
- understanding highly non-perturbative/collective effects like phase transitions requires the application of numerical techniques – lattice QCD

40 years of lattice QCD thermodynamics:

the first direct evidence for the existence of a thermal phase transition in SU(Nc) gauge theories from lattice calculations has been presented at a conference in Bielefeld

Statistical Mechanics of Quarks and Hadrons
ZIF, Bielefeld, August 1980 (organizer H. Satz)

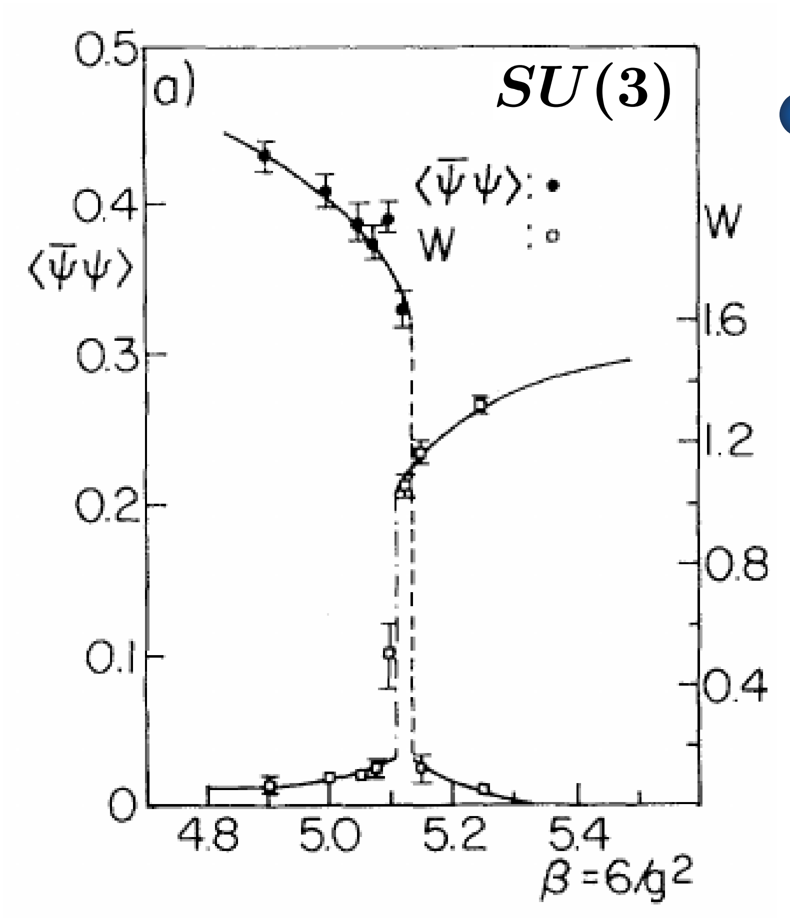
B. Svetitsky and L. McLerran, PLB 98 (1981)
J. Kuti, J. Polonyi and K. Szlachanyi PLB 98 (1981)

2020/21: speed increased by 10^{12}, i.e. a factor 2 every year

1980/81:
first lattice calculations of the EoS

J. Engels, FK, I. Montvay, H. Satz, PLB 101 (1981)

Deconfinement & Chiral Symmetry Restoration

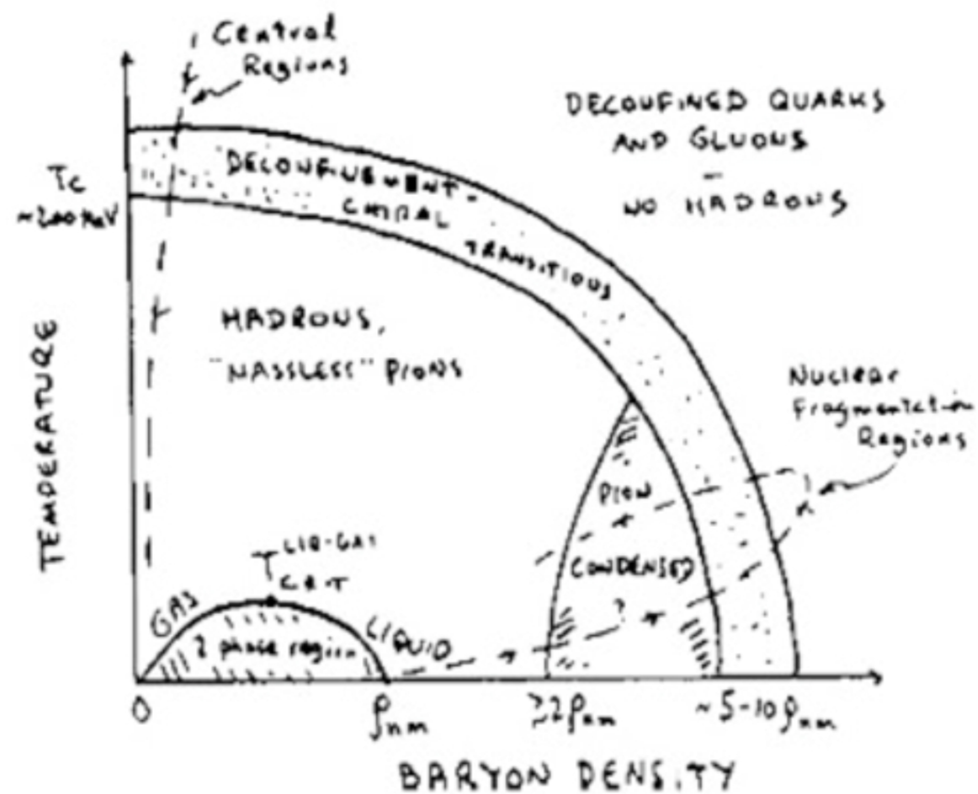


W: Polyakov loop expectation value

$\langle\bar\psi\psi\rangle$: chiral condensate

- deconfinement and chiral symmetry restoration are closely related in QCD

J. Kogut et al., PRL 50 (1983) 393



Gordon Baym: Long Range Plan 1983

The chiral PHASE TRANSITION temperature

R. D. Pisarski, F. Wilczek,
Remarks on the chiral phase transition in chromodynamics,
Phys. Rev. D 29 (1984) 338(R)

Abstract:
The phase transition restoring chiral symmetry at finite temperatures is considered in a linear σ model. For three or more massless flavors, the perturbative ϵ expansion predicts the phase transition is of first order.
At high temperatures, the UA(1) symmetry will also be effectively restored.

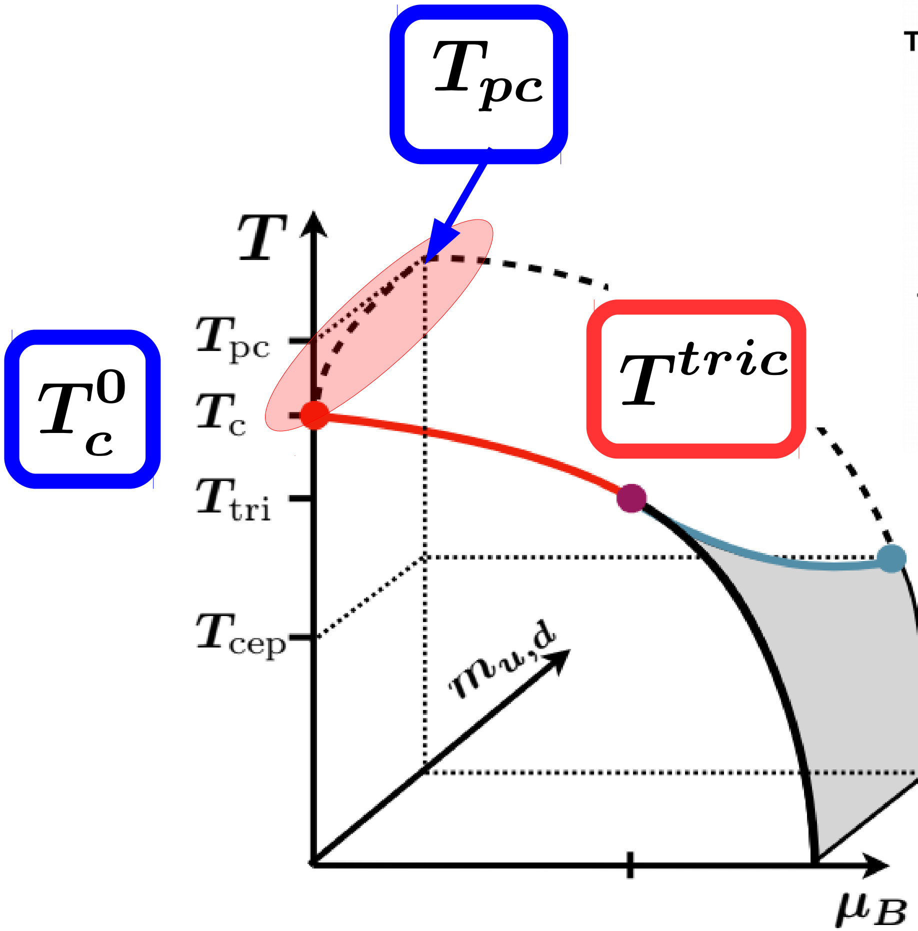
– since 35 years it is understood that critical behavior in strong-interaction matter is due to **chiral symmetry restoration**

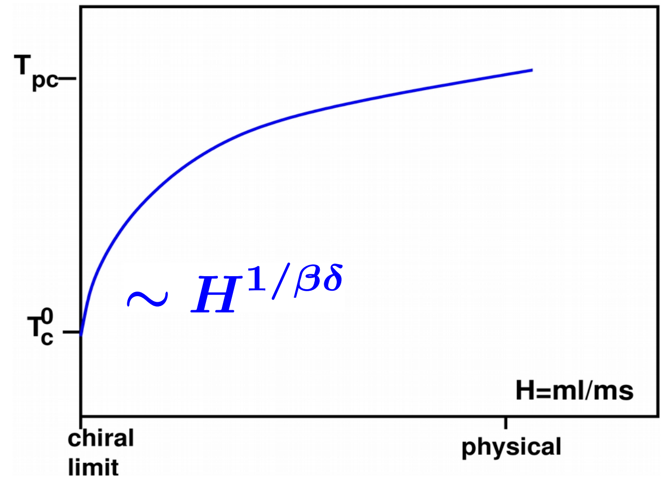
– the **chiral phase transition** of QCD at vanishing values of the light quark masses puts severe constraints on the structure of the phase diagram at physical values of the quark masses

– **deconfinement transition ?**

Phases of strong-interaction matter

determination of T_c^0 puts an upper limit on T^{CEP}





hierarchy of (pseudo-) critical temperatures

$$T_{pc} > T_c^0 > T_{tric} > T_{CEP}$$

Random Matrix Model A. Halasz, A.D. Jackson, R.E. Shrock, M.A. Stephanov, J.J.,M. Verbaarschot, Phys. Rev. D58 (1998) 096007

QCD M. Stephanov, Phys. Rev. D73 (2006) 094508

NJL M. Buballa, S. Carignano, Phys. Lett. B791 (2019) 361

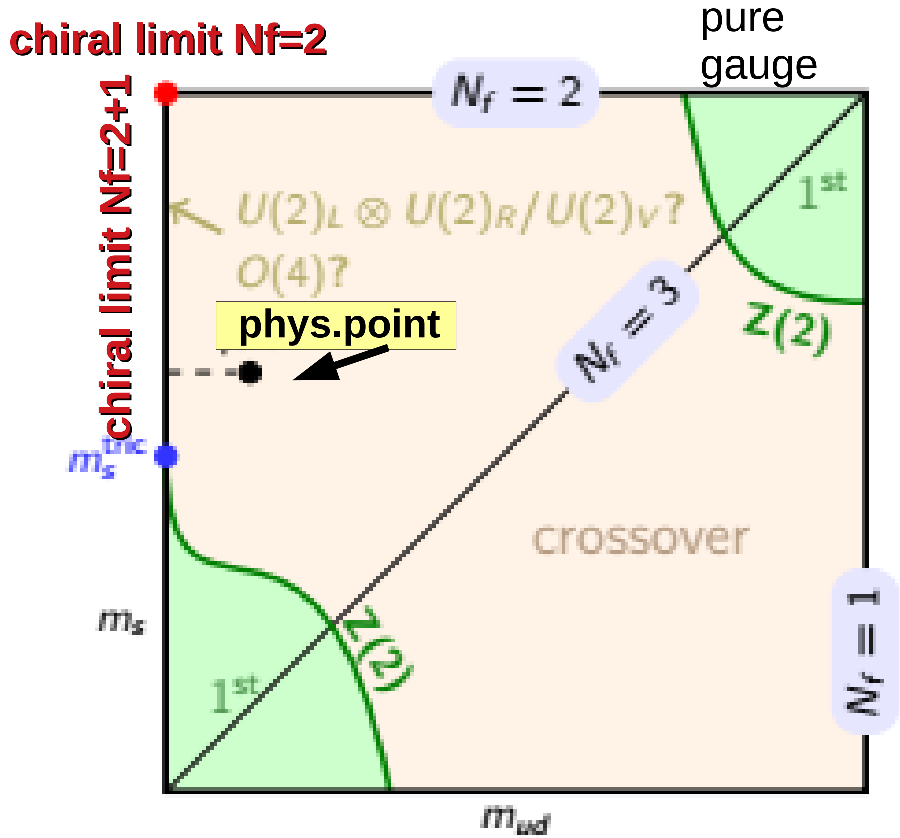
Phases of strong-interaction matter

– volume dependence of chiral susceptibility puts strong bounds on the possible existence of first order transitions in the QCD phase diagram

– chiral order parameter: $M = \frac{2}{f_K^4}\left[m_s\langle\bar{\psi}\psi\rangle_l - m_l\langle\bar{\psi}\psi\rangle_s\right]$

$$m_l = (m_u + m_d)/2$$

– chiral susceptibility: $\chi_M = m_s\left(\frac{\partial M}{\partial m_u} + \frac{\partial M}{\partial m_d}\right)$



2[nd] order chiral phase transition in (2+1)-flavor QCD?

1[st] order chiral phase transition in 3-flavor QCD?

Phases of strong-interaction matter

– volume dependence of chiral susceptibility puts strong bounds on the possible existence of first order transitions in the QCD phase diagram

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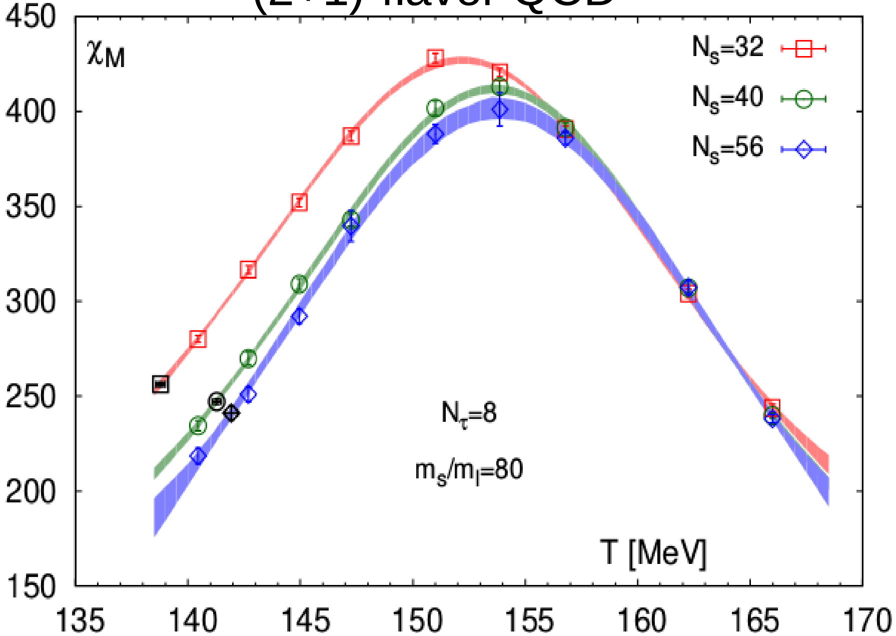
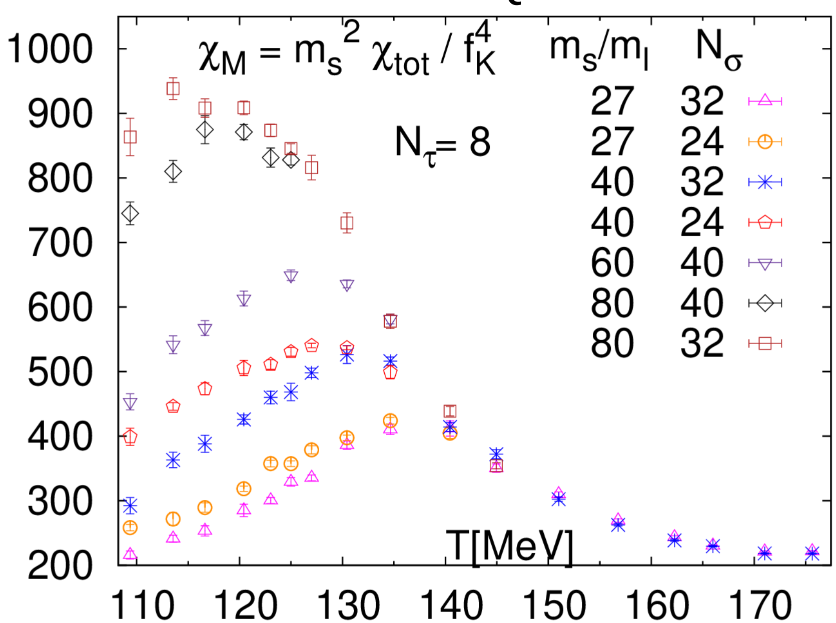
(2+1)-flavor QCD



HotQCD, PRL 123 (2019) 062002

3-flavor QCD



Sipaz Sharma et al., Lattice 2021

– no hint for 1st order transition down to $m_\pi = 55$ and 80 MeV, respectively.

– no 1st order phase transition for : $N_f < 6$ F. Cuteri et al., arXiv:2107.12739

Critical behavior in QCD

- close to the chiral limit thermodynamics in the vicinity of the QCD transition(s) is controlled by a universal scaling function

singular regular

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(V,T,\vec{\mu}) = -h^{(2-\alpha)/\beta\delta} f_f(t/h^{1/\beta\delta}) - f_r(V,T,\vec{\mu})$$

$$t \sim \frac{T-T_c}{T_c} + \kappa_2 \left(\frac{\mu_q}{T}\right)^2$$

$$h \sim \frac{m_q}{T_c}$$

Pseudo-critical temperatures

response functions
2nd order cumulants

- magnetic — mixed — thermal

$$\frac{\partial^2 \ln Z}{\partial h^2} \qquad \frac{\partial^2 \ln Z}{\partial h\, \partial t} \qquad \frac{\partial^2 \ln Z}{\partial t^2} \sim \frac{\partial^4 \ln Z}{\partial \mu_q^4}$$

$$\sim \left(\frac{m_l}{T_c}\right)^{1/\delta - 1} \qquad \sim \left(\frac{m_l}{T_c}\right)^{(\beta-1)/\beta\delta} \qquad \sim \left(\frac{m_l}{T_c}\right)^{-\alpha/\beta\delta}$$

$$\sim -0.79 \qquad \sim -0.34 \qquad \sim +0.11$$

O(4) critical exponents

$\alpha = -0.21$

$\beta = 0.38$

$\delta = 4.82$

divergence: strong — moderate — none

Scaling in the thermodynamic (infinite volume) limit
– approaching the chiral limit –

some definitions

$$z = \frac{t}{h^{1/\beta\delta}}$$
$$t \equiv \frac{1}{t_0}\frac{T-T_c}{T_c}$$
$$h = \frac{1}{h_0}H$$
$$H \equiv \frac{m_l}{m_s}$$
$$z_0 = h_0^{1/\beta\delta}/t_0$$

– order parameter M and its susceptibility

$$M = h^{1/\delta} f_G(z) + f_{sub}(T,H)$$
$$\chi_M = h_0^{-1} h^{1/\delta-1} f_\chi(z) + \tilde{f}_{sub}(T,H)$$

for ANY fixed z:

$$T_{pc}(H) = T_c^0\left(1+\frac{z}{z_0}H^{1/\beta\delta}\right) + \text{sub leading}$$

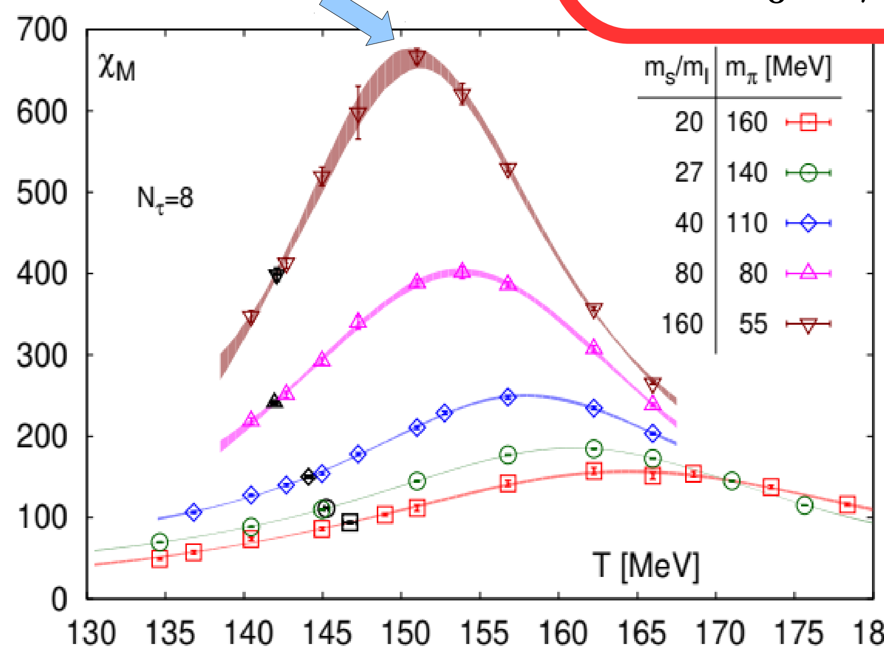
conventional steps to determine T_c^0

– choose a characteristic feature of χ_M
→ the maximum χ_M^{max}

– in the scaling regime this is located at $z_p \simeq 1.5$

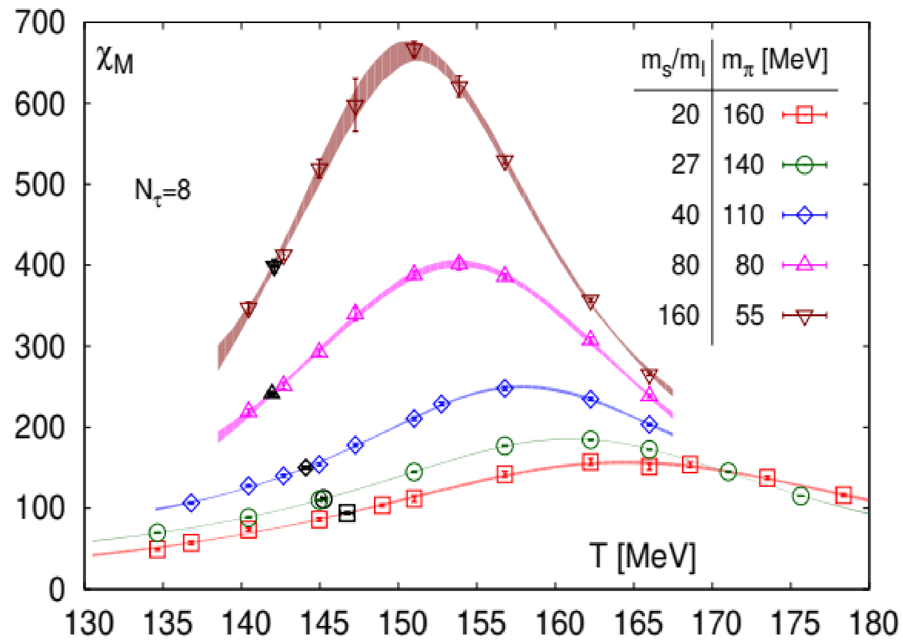
– using the scaling ansatz for $T_{pc}(H)$ allows to extract T_c^0

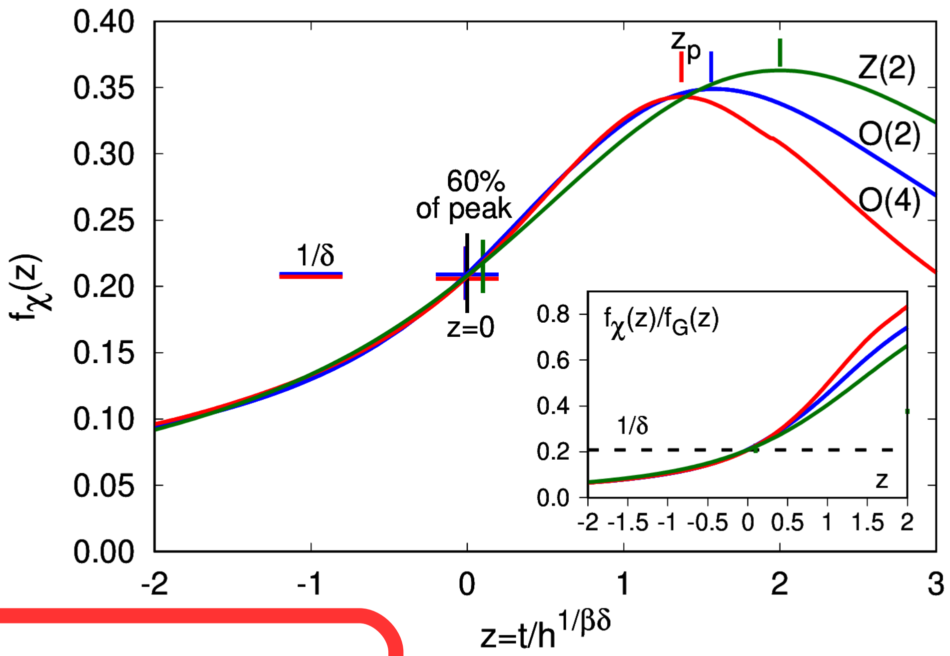
$\chi_M^{max} \sim H^{1/\delta-1}$



A. Lahiri et al, QM 2018, arXiv:1807.05727
H.T. Ding et al, arXiv:1903.04801

Chiral PHASE TRANSITION temperature





$T_{60} \sim T_\delta$ temperature at which χ_M reaches 60% of its maximal value

estimators for T_c^0

$$\frac{H\chi_M}{M} = \frac{f_\chi(z)}{f_G(z)} + \text{ regular}$$

$$= \begin{cases} 1/\delta & , \ z = 0 \quad \Rightarrow T_\delta \\ \sim 0.5 & , \ z = z_p \end{cases}$$

Finite size scaling functions of the 3-d, O(4) spin model

$$M = h^{1/\delta} f_G(z, z_L) + f_{sub}(T,H,L)$$

$$\chi_M = h_0^{-1} h^{1/\delta-1} f_\chi(z, z_L) + \tilde{f}_{sub}(T,H,L)$$

$$z_L = \frac{1}{L h^{\nu_c}}$$

$$z = \frac{t}{h^{1/\beta\delta}}$$

$$\nu_c = \nu/\beta\delta = (0.5 - 0.6)$$

a universal ratio for each z:

$$\boxed{\frac{H\chi_M}{M} = \frac{f_\chi(z,z_L)}{f_G(z,z_L)} + \text{sub leading}}$$

$$\lim_{L\to\infty} \left(\frac{H\chi_M}{M}\right)_{z=0} = \frac{1}{\delta}$$

volume dependence controlled by $z_L \sim 1/(m_\pi^{2\nu_c} L)\ ,\ 2\nu_c \simeq 1$

define $z_\delta(z_L)$ as the value z for given z_L at which $\left(\frac{H\chi_M}{M}\right)_{z_\delta(z_L)} = \frac{1}{\delta}$

$$T_\delta(H,L) = T_c^0 \left(1 + \frac{z_\delta(z_L)}{z_0} H^{1/\beta\delta}\right) + \text{sub leading}$$

$$z_\delta(0) = 0$$

$z_\delta \simeq 0 \quad \Rightarrow$ weak H-dependence of T_δ even at finite H and/or L
– almost perfect estimator for T_c in the limit $H \to 0\ ,\ L \to \infty$

Finite size scaling functions of the 3-d, O(4) spin model

$$V \equiv L^3 \equiv (N_\sigma a)^3$$

$$M = h^{1/\delta} f_G(z, z_L) + f_{sub}(T, H, L)$$
$$\chi_M = h_0^{-1} h^{1/\delta - 1} f_\chi(z, z_L) + \tilde{f}_{sub}(T, H, L)$$

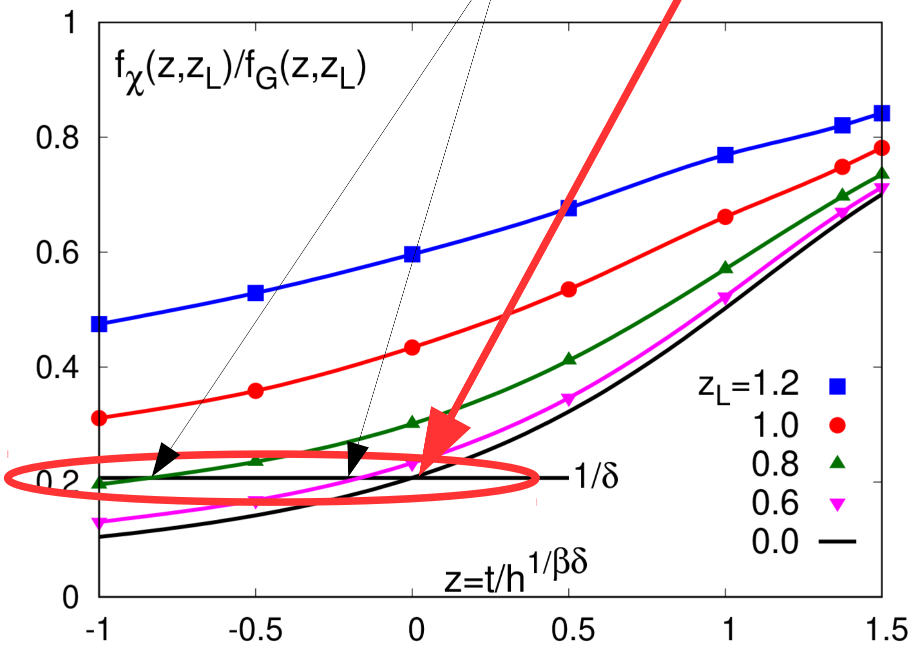
$$\frac{H\chi_M}{M} = \frac{f_\chi(z, z_L)}{f_G(z, z_L)} + \text{sub leading}$$

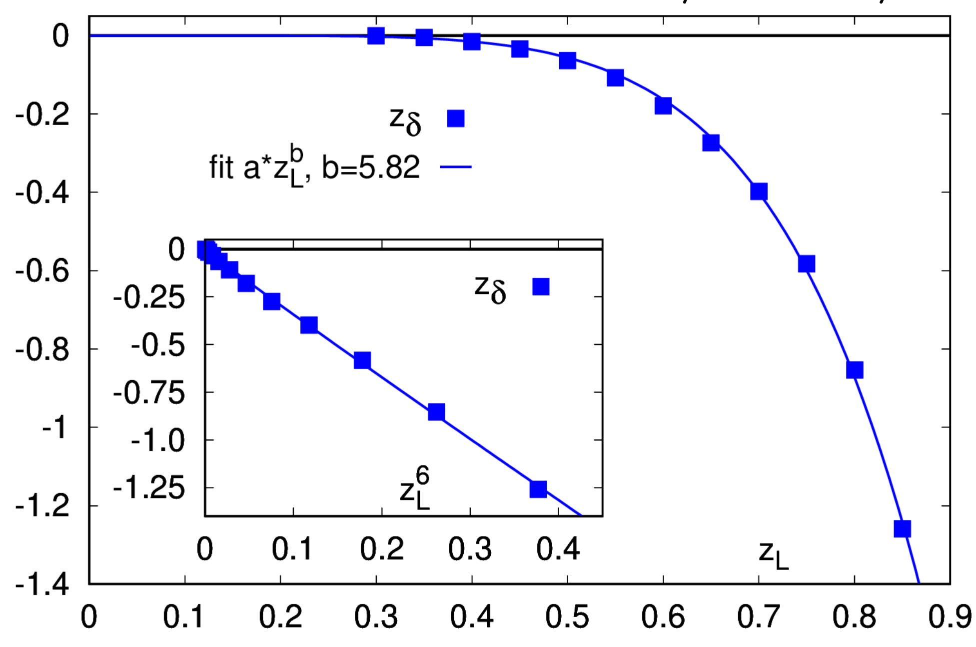
quark mass dependence arises only as a finite volume effect (+s.l.)

$$T_\delta(H, L) = T_c^0 \left(1 + \frac{z_\delta(z_L)}{z_0} H^{1/\beta\delta}\right)$$

$$\lim_{L\to\infty} T_\delta(L) = T_c^0$$

finite volume effects $\sim 1/V^2 \equiv 1/L^6$





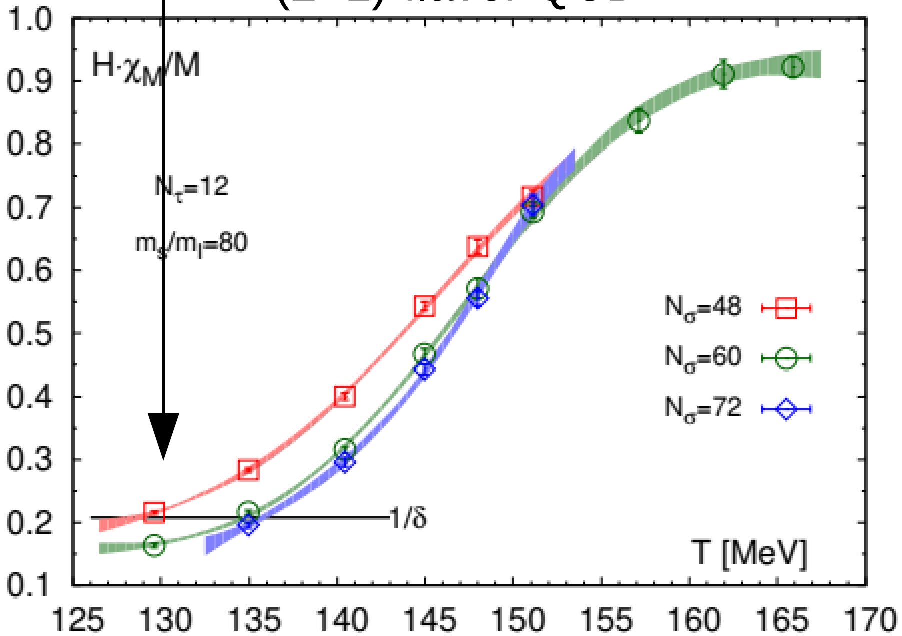
Chiral PHASE TRANSITION in (2+1)&3-flavor QCD

use a novel observable for the determination of the chiral PHASE TRANSITION TEMPERATURE, which in the infinite volume limit correspond to $z \simeq 0$, i.e. in the scaling regime they have almost no quark mass dependence

$$T_\delta(H,L) = T_c^0 \left(1 + \frac{z_\delta(z_L)}{z_0} H^{1/\beta\delta}\right) + \text{sub leading}$$

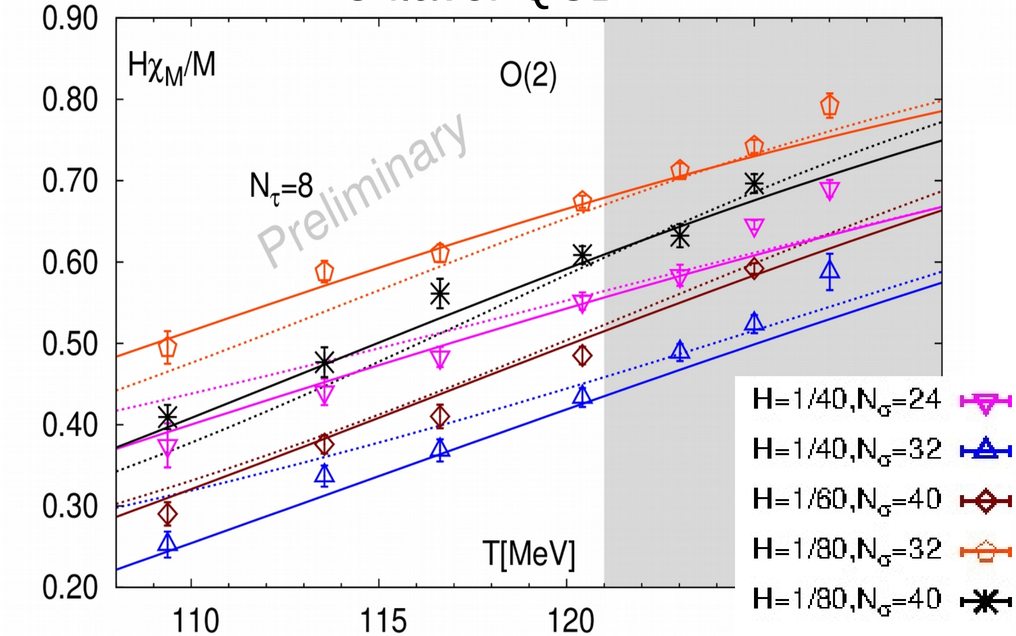
$$\frac{H\chi_M}{M} = \frac{1}{\delta} \Rightarrow T_\delta$$

(2+1)-flavor QCD



HotQCD, PRL 123 (2019) 062002

3-flavor QCD



Sipaz Sharma et al., Lattice 2021

Finite size & and quark mass scaling

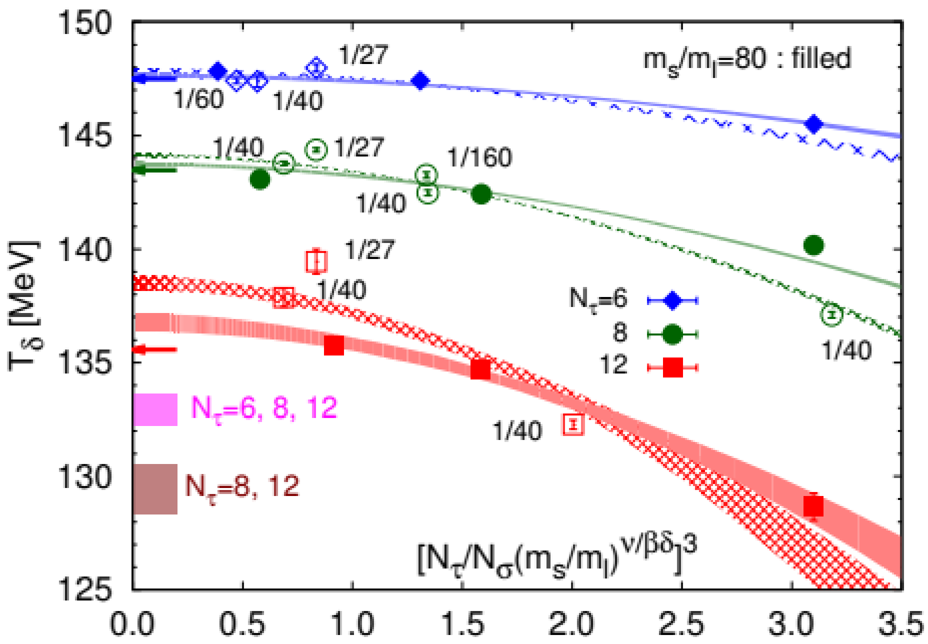
$$T_\delta(H,L) = T_c^0\left(1+\frac{z_\delta(z_L)}{z_0}H^{1/\beta\delta}\right) + \text{sub leading}$$

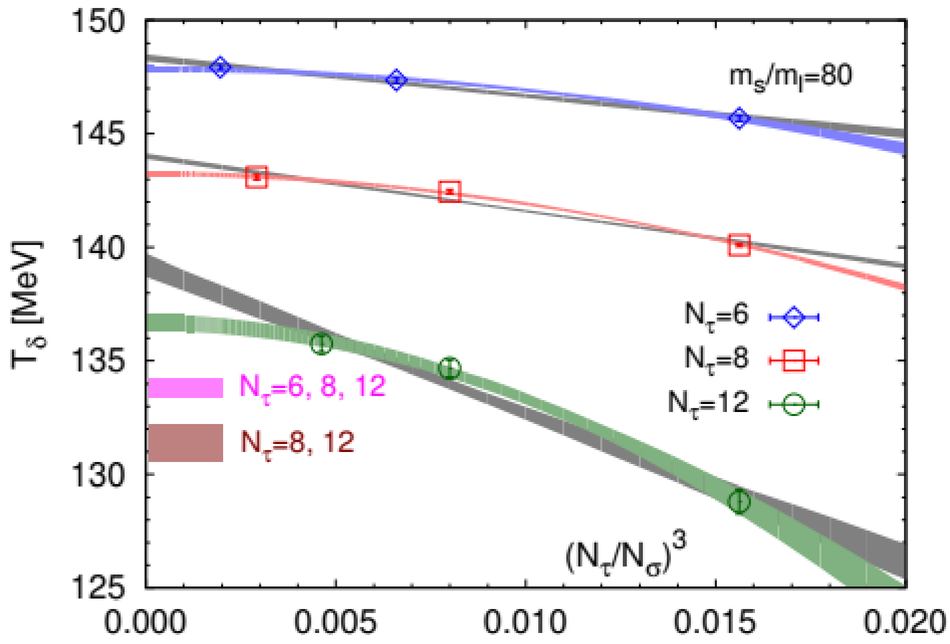
$$\left(T_\delta(H,L)/T_c^0 - 1\right) H^{-1/\beta\delta} - c_r H^{1-1/\delta} = \frac{z_\delta(z_L)}{z_0}$$

leading regular term

$$z_L = z_{L,0}\left(\frac{m_s}{m_l}\right)^{\nu_c}\frac{N_\tau}{N_\sigma}$$

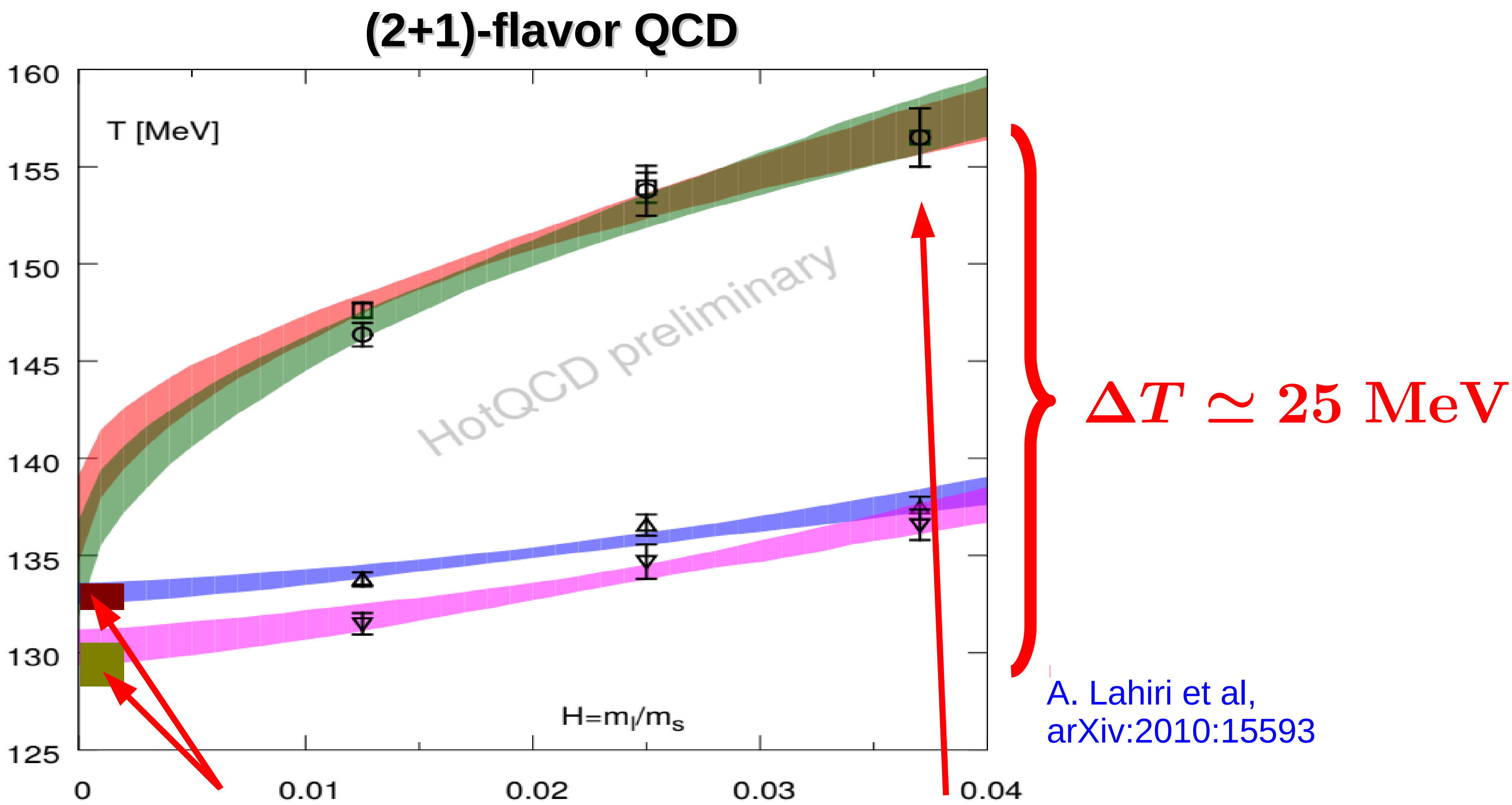
$$\nu_c = \nu/\beta\delta$$





$z_\delta(z_L)$ for O(4) from
J. Engels, FK, Phys. Rev. D90 (2014) 014501

Chiral PHASE TRANSITION temperature



A. Lahiri et al,
arXiv:2010:15593

chiral limit extrapolations

physical masses

$$T_c^0 = 132^{+3}_{-6} \text{ MeV}$$

H.-T. Ding et al [HotQCD],
arXiv:1903.04801

$$T_{pc}^{phys} = (156.5 \pm 1.5) \text{ MeV}$$

A. Bazavov et al [HotQCD],
arXiv:1812.08235

The chiral PHASE TRANSITION temperature

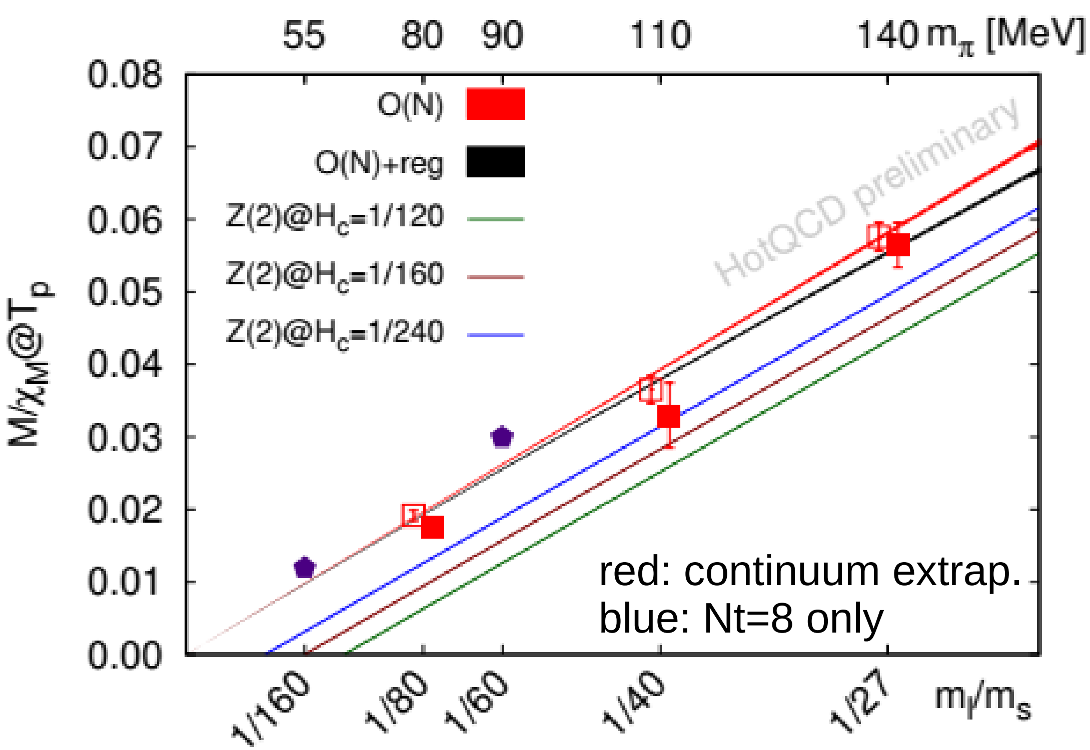
– evidence for a 2nd order transition in the chiral limit–

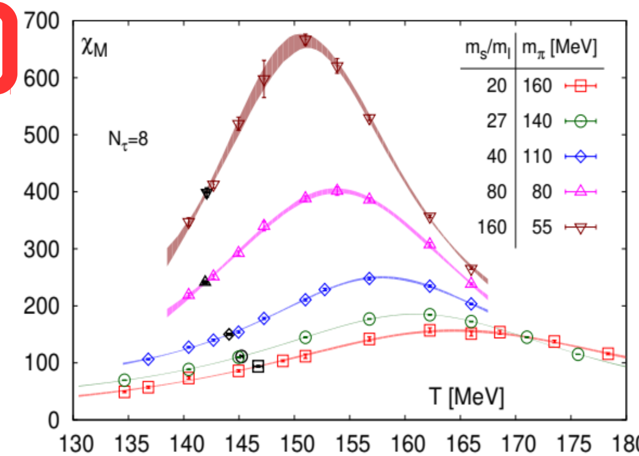
in the thermodynamic limit: suppose there occurs a 1st order transition for $H < H_c$

$$M(T,H) \sim (H-H_c)^{1/\delta} f_G(z)$$ (M is "almost" an order parameter)

$$\chi(T,H) \sim (H-H_c)^{1/\delta-1} f_\chi(z) + \ldots$$

for ANY fixed z: $\frac{M}{\chi_M} \sim (H-H_c)\,\frac{f_G(z)}{f_\chi(z)}$ → $\boxed{\text{bound on } H_c}$





χ_M would diverge already for non-zero H_c

A. Lahiri et al, QCD&HRG, Wroclaw 2020, arXiv:2010.15593

Chiral observables in QCD

– chiral condensate: $\langle\bar{\psi}\psi\rangle_q = \frac{\partial P/T}{\partial m_q/T}$, $\langle\bar{\psi}\psi\rangle_l = (\langle\bar{\psi}\psi\rangle_u + \langle\bar{\psi}\psi\rangle_d)/2$

– chiral order parameter: $M = \frac{2}{f_K^4}\left[m_s\langle\bar{\psi}\psi\rangle_l - m_l\langle\bar{\psi}\psi\rangle_s\right]$

$$m_l = (m_u + m_d)/2$$

– chiral susceptibility: $\chi_M = m_s\left(\frac{\partial M}{\partial m_u} + \frac{\partial M}{\partial m_d}\right)$ **magnetic**

– mixed chiral susceptibility: $\chi_t = T\frac{\partial M}{\partial T} \sim T\frac{\partial^2 P/T^4}{\partial T\partial m_q}$ **mixed**

$$\sim T^2\frac{\partial^3 P/T^4}{\partial m_q\partial\mu_X^2}\bigg|_{\mu_X=0}$$

– conserved charge fluctuations: $\chi_X = T^2\frac{\partial^2 P/T^4}{\partial\mu_X^2}\bigg|_{\mu_X=0}$ **thermal** $X = B,\ S, \ldots$

Pseudo-critical temperature at non-zero μ_B

universal scaling relations determine curvature of the crossover line

– μ_B-dependent shift of maxima in susceptibilities

$$M(T,\mu_B) = M(T,0) + \frac{\partial M}{\partial T}(T-T_c) + \frac{1}{2}\frac{\partial^2 M}{\partial(\mu_B/T)^2}\bigg|_{\mu_B=0}\left(\frac{\mu_B}{T}\right)^2 + \ldots$$

$$\frac{\partial^2 M(T,\mu_B)}{\partial T^2} = 0 \;:\; T_{pc}(\mu_B) = T_{pc}\left(1 - \kappa_2\left(\frac{\mu_B}{T_c}\right)^2 - \kappa_4\left(\frac{\mu_B}{T_c}\right)^4 + \ldots\right)$$

– universality relations also relate derivatives with respect to T and μ_B

$$\frac{1}{VT^3}\ln Z(V,T,\vec{\mu}) \sim -h^{(2-\alpha)/\beta\delta} f_f(t/h^{1/\beta\delta})$$

$$\boxed{h \sim \frac{m_q}{T_c} \;\; , t \sim \frac{T-T_c}{T_c} + \kappa_2\left(\frac{\mu_B}{T}\right)^2} \quad \Leftrightarrow \quad \boxed{\frac{\partial^2}{\partial(\mu_B/T)^2} \simeq \frac{\partial}{\partial T}}$$

FK et al., arXiv:1009.5211

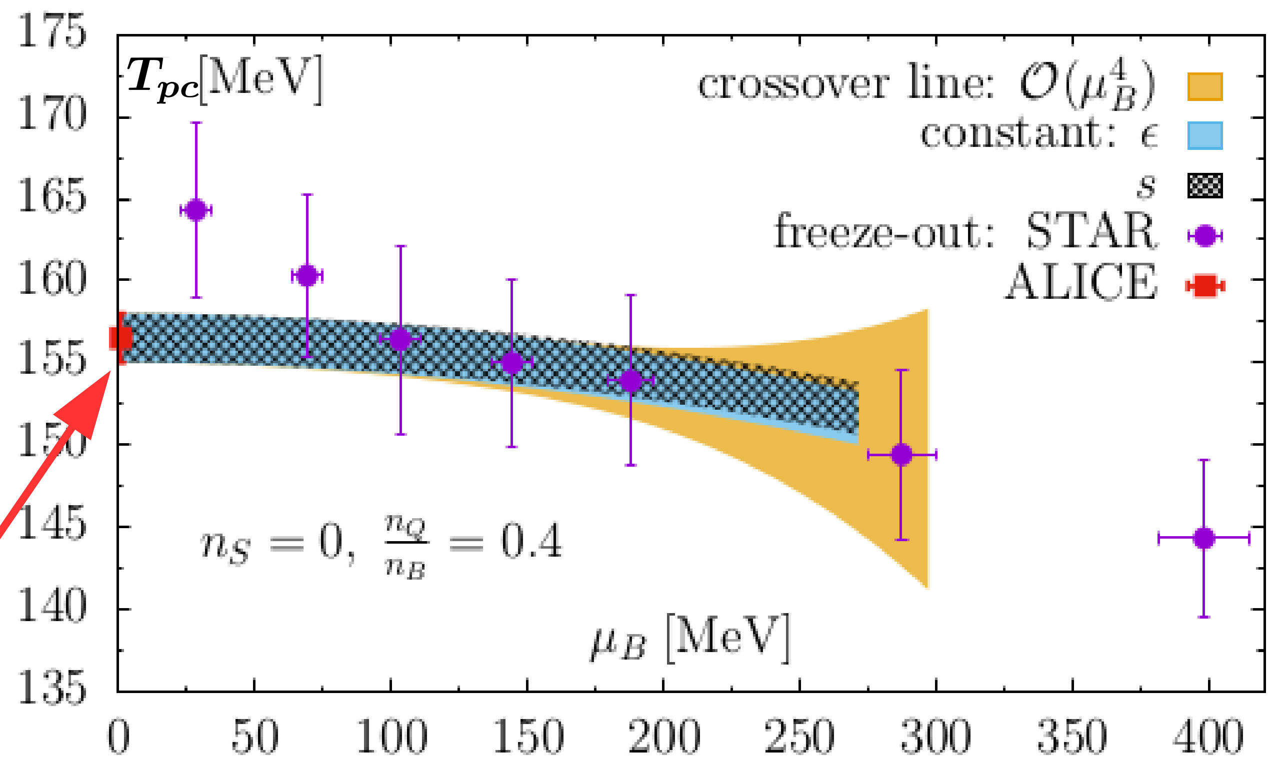
$$\Rightarrow \quad \kappa_2 \simeq \frac{T^2\partial^2 M/\partial\mu_B^2}{2T\partial M/\partial T}\bigg|_{T_c(0)}$$

numerator and denominator diverge in chiral limit;
$\Rightarrow$ regular terms drop out

Phases of strong-interaction matter

$$T_{pc}(\mu_B) = T_{pc}\left(1 - \kappa_2\left(\frac{\mu_B}{T_c}\right)^2 - \kappa_4\left(\frac{\mu_B}{T_c}\right)^4 + \ldots\right)$$

phase diagram at physical values of the quark masses



STAR: arXiv:1701.07065

A. Andronic et al., Nature 561 (2018) 321

$$T_{pc} = (156.5 \pm 1.5)\ \mathrm{MeV}$$

$$\kappa_2 = 0.012(4)$$
$$\kappa_4 = 0.000(4)$$

A. Bazavov et al. [HotQCD], Phys. Lett. B795, 15 (2019), arXiv:1812.08235

$$T_{pc} = (158.0 \pm 0.6)\ \mathrm{MeV}$$

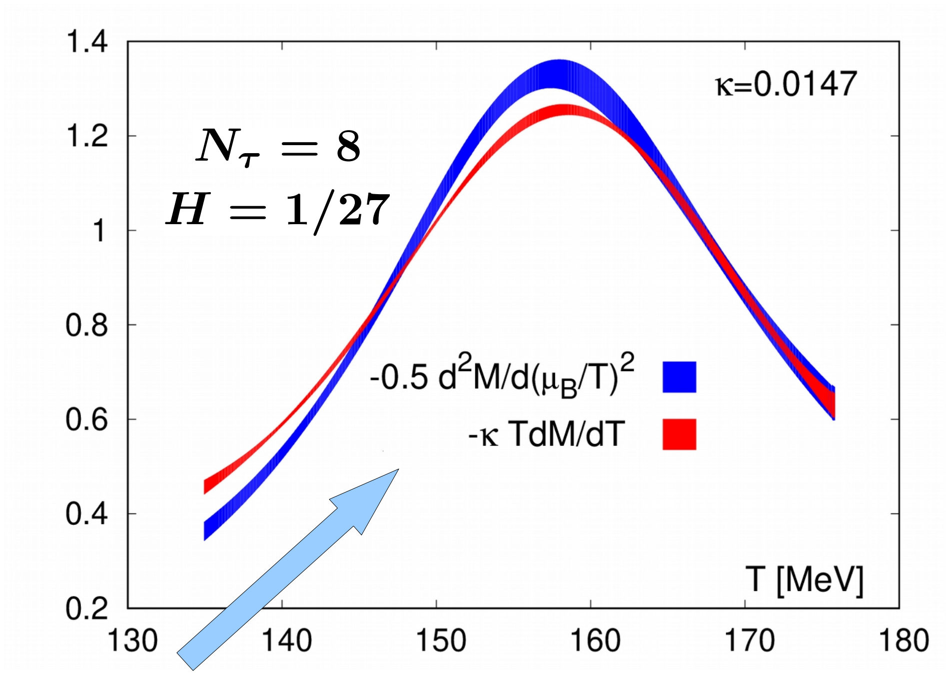
$$\kappa_2 = 0.0153(18)$$
$$\kappa_4 = 0.00032(67)$$

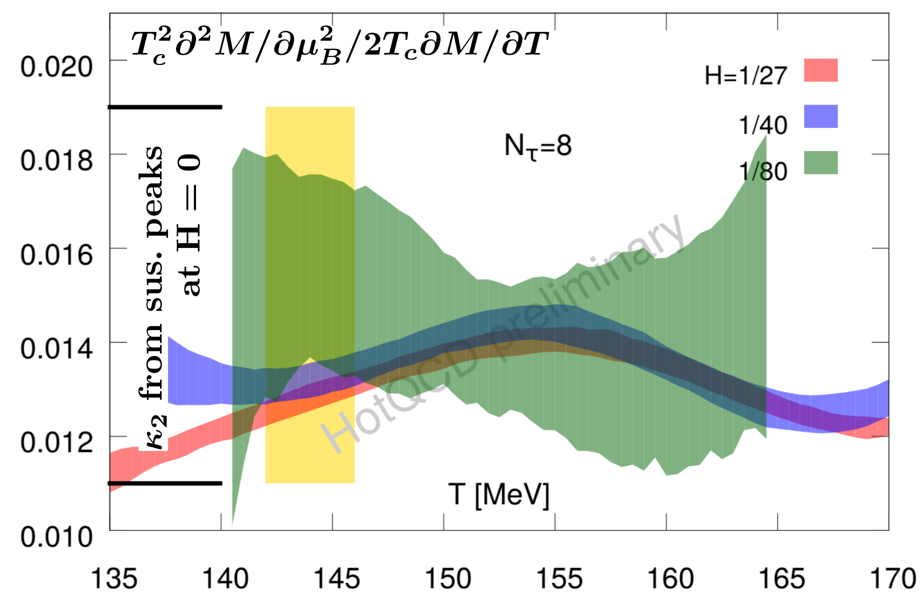
S. Borsanyi, et al, arXiv:2002.02821

Critical behavior and higher order cumulants
– towards the chiral limit –

critical behavior in chiral observables: derivatives of the chiral condensate

$$M(T,\mu_B) = M(T,0) + \frac{\partial M}{\partial T}(T-T_c) + \frac{1}{2}\frac{\partial^2 M}{\partial(\mu_B/T)^2}\bigg|_{\mu_B=0}\left(\frac{\mu_B}{T}\right)^2 + \ldots$$





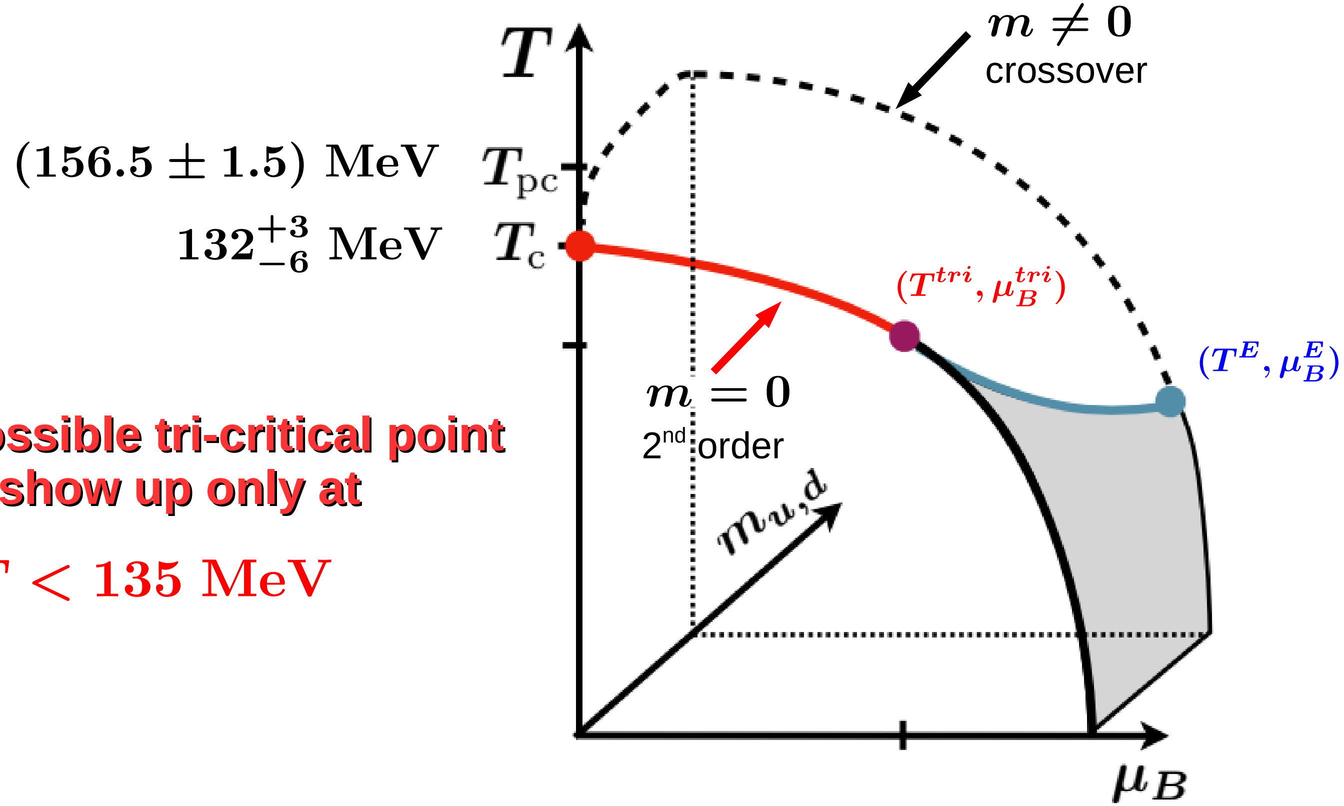
Mugdha Sarkar (HotQCD), Lattice 2021

$$\kappa_2 \simeq \frac{T^2\partial^2 M/\partial\mu_B^2}{2T\partial M/\partial T}\bigg|_{T_c(0)}$$

$$t \sim \frac{T-T_c}{T_c} + \kappa_2(H)\left(\frac{\mu_B}{T}\right)^2 , \; H = m_l/m_s$$

curvature of crossover line only mildly (not) dependent on H

Crossover, chiral phase transition at $\mu_B = 0$ and the (tri)-critical point at $\mu_B > 0$



a possible tri-critical point will show up only at

$T < 135$ MeV

the critical point is likely to be located at

$T < 130$ MeV

Note: $T_c(\mu_B)$ drops by about 10 MeV between

$\mu_B = 0$ and 400 MeV

Energy-like observables in the chiral limit

– **does the Polyakov loop play any role in characterizing deconfinement in the chiral limit ?**

– heavy quark free energy $F_q(T,H) = -T\ln\langle L\rangle = -\frac{T}{2}\lim_{|\vec{x}-\vec{y}|\to\infty}\ln\langle P^\dagger_{\vec{x}}P_{\vec{y}}\rangle$

– **Polyakov loop is blind to chiral rotations $\Rightarrow$ energy like observable**

– expect: $F_q(T,0) \sim a_\pm |(T-T_c)/T_c|^{1-\alpha} + reg$

$$T_c \frac{\partial F_q(T,H)/T}{\partial T} = T_c \frac{1}{\langle P\rangle}\frac{\partial\langle P\rangle}{\partial T}$$

$$\sim -AH^{(-\alpha)/\beta\delta} f''_f(z) + reg$$

$\alpha < 0 \;\Rightarrow$ not divergent; cusp

$$\frac{\partial F_q(T,H)/T}{\partial H} =$$

$$-\frac{1}{\langle L\rangle}\frac{\partial\langle P\rangle}{\partial H} \sim -AH^{(\beta-1)/\beta\delta} f'_G(z) + reg$$

divergent

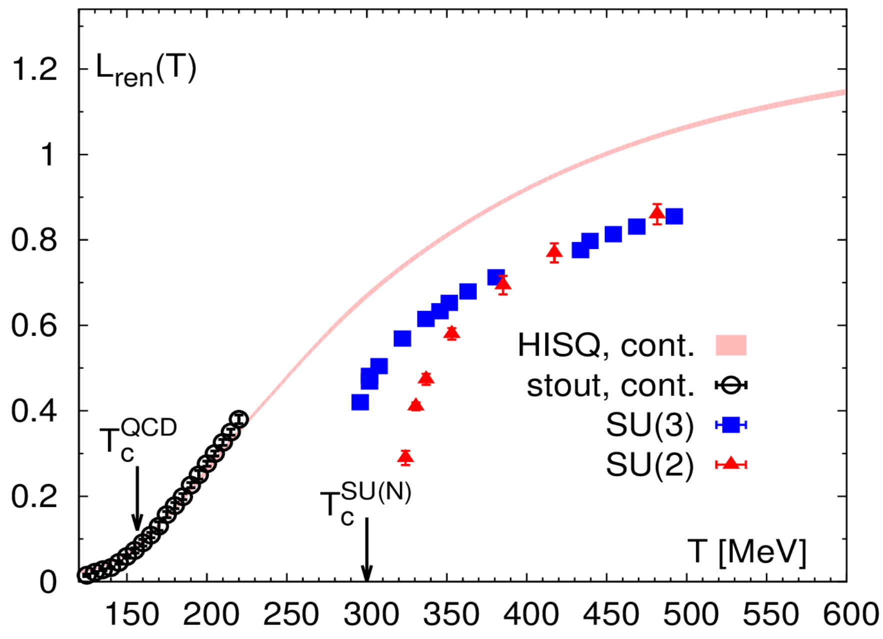
Energy-like observables in the chiral limit

– **does the Polyakov loop play any role in characterizing deconfinement in the chiral limit ?**

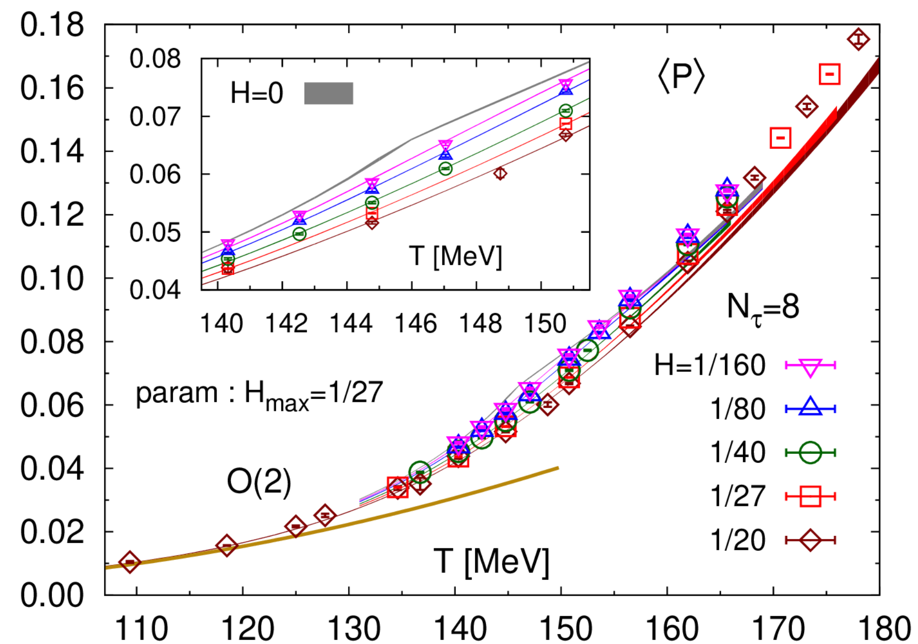
– heavy quark free energy $F_q(T,H) = -T\ln\langle L\rangle = -\frac{T}{2}\lim_{|\vec{x}-\vec{y}|\to\infty}\ln\langle P^\dagger_{\vec{x}}P_{\vec{y}}\rangle$

– **Polyakov loop is blind to chiral rotations ⇒ energy like observable**

– expect: $F_q(T,0) \sim a_\pm |(T-T_c)/T_c|^{1-\alpha} + reg$

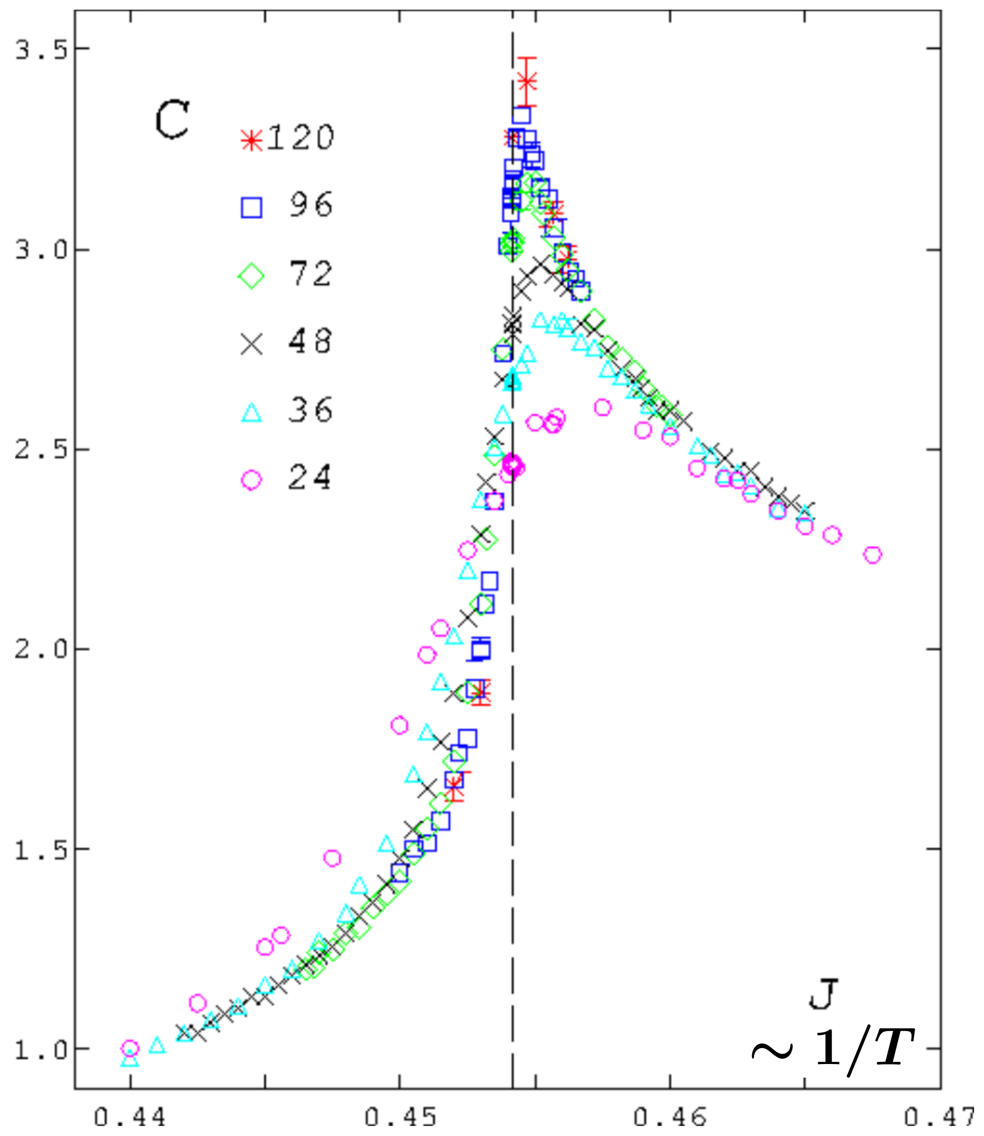


P. Petreczky, arXiv: 2011.01466



D.A. Clarke et al, PRD 103 (2021) L011501

Energy-like observables in O(N) spin models

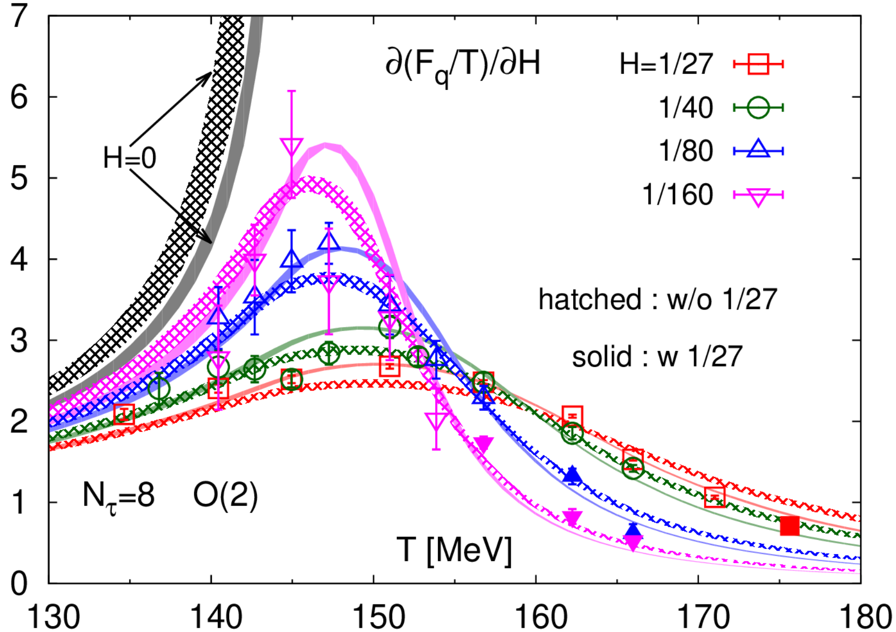


Specific heat in the 3d, O(2) spin model,
A. Cucchieri et al, J. Phys A 35 (2002) 6517

Energy-like observables in the chiral limit

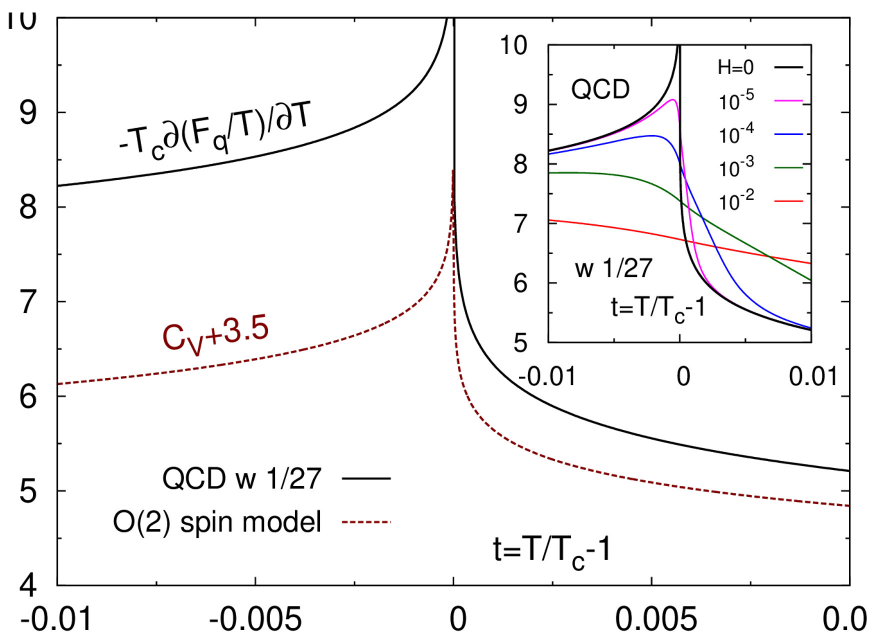
– Polyakov loop is blind to chiral rotations ⇨ energy like observable

D.A. Clarke et al, arXiv:2008.11678



– H-derivative is a mixed susceptibility

$$\frac{\partial F_q(T,H)/T}{\partial H} = -\frac{1}{\langle P\rangle}\frac{\partial \langle P\rangle}{\partial H} \sim -AH^{(\beta-1)/\beta\delta} f'_G(z) + reg$$

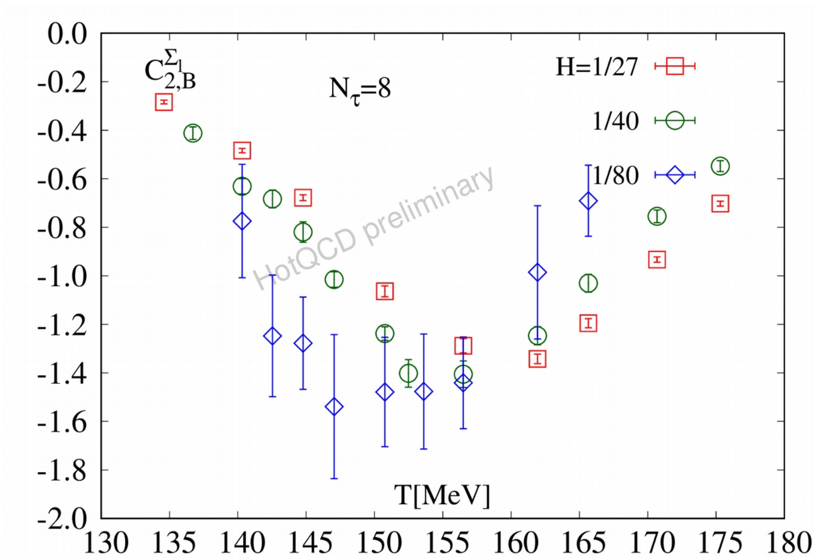


– T-derivative behaves like specific heat in O(N) spin models

$$T_c \frac{\partial F_q(T,H)/T}{\partial T} \sim -AH^{(-\alpha)/\beta\delta} f''_f(z) + reg$$

Specific heat in the 3d, O(2) spin model,
A. Cucchieri et al, J. Phys A 35 (2002) 6517

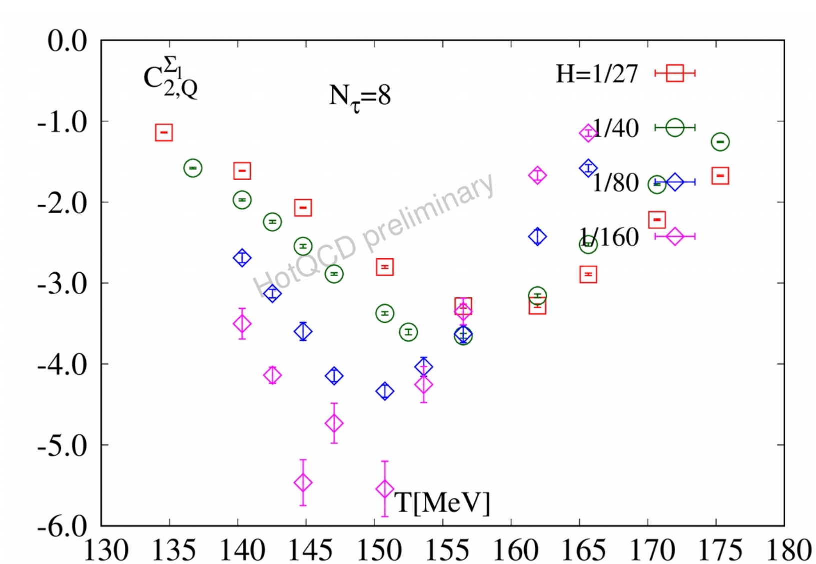
Critical behavior in conserved charge cumulants
– Energy-like observables in the chiral limit –



$$C^{\Sigma}_{2,B} \sim \frac{\partial \langle \bar{\psi}\psi \rangle_l}{\partial (\mu_B/T)^2} = \frac{\partial^2 P/T^4}{\partial m_l \partial (\mu_B/T)^2}$$

$$\sim -AH^{(\beta-1)/\beta\delta} f'_G(z) + reg$$

mixed derivative;
divergent in chiral limit



$$C^{\Sigma}_{2,Q} \sim \frac{\partial \langle \bar{\psi}\psi \rangle_l}{\partial (\mu_Q/T)^2} = \frac{\partial^2 P/T^4}{\partial m_l \partial (\mu_Q/T)^2}$$

$$\sim -AH^{(\beta-1)/\beta\delta} f'_G(z) + reg$$

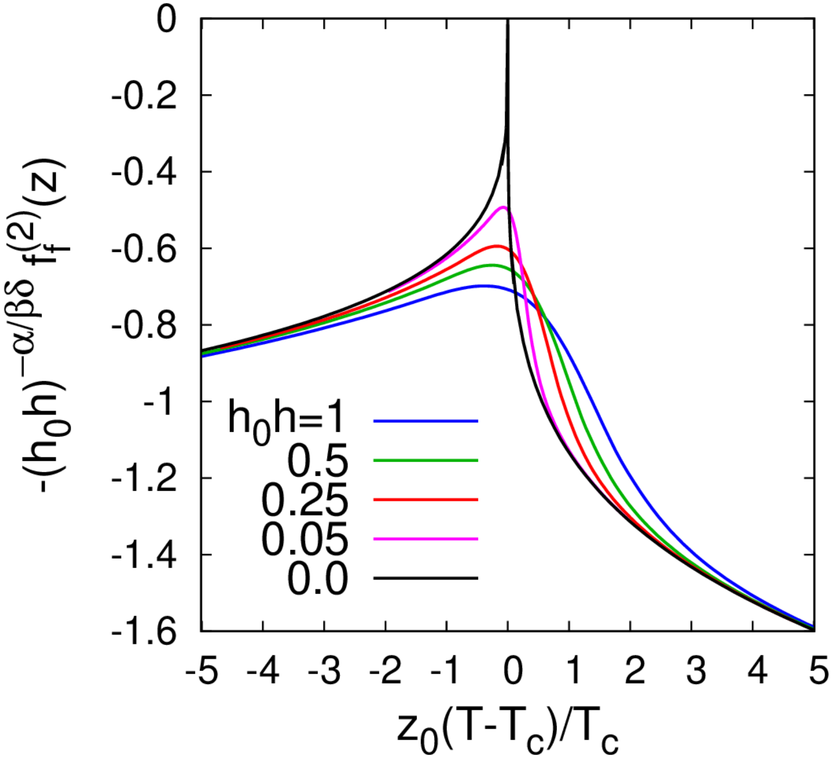
mixed derivative;
divergent in chiral limit

Critical behavior in conserved charge cumulants – Energy-like observables in the chiral limit –

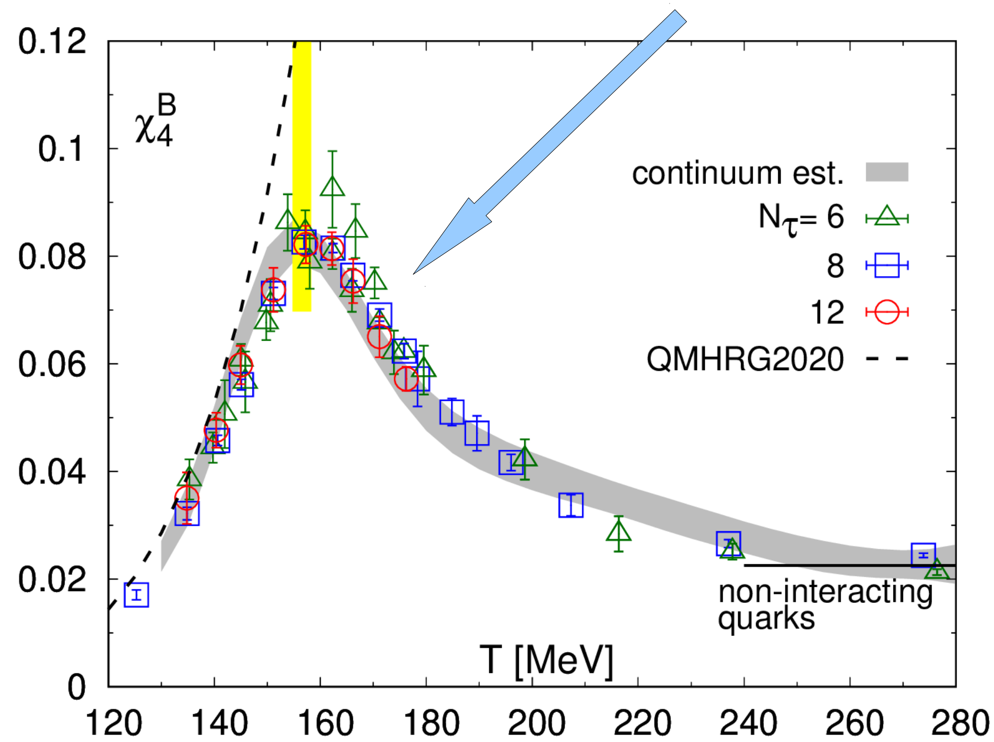
– conserved charge cumulants are energy like observables

– fourth order cumulants will develop a cusp in the chiral limit, which becomes self-evident only at VERY small quark masses

huge deviations from HRG = deconfinement !?

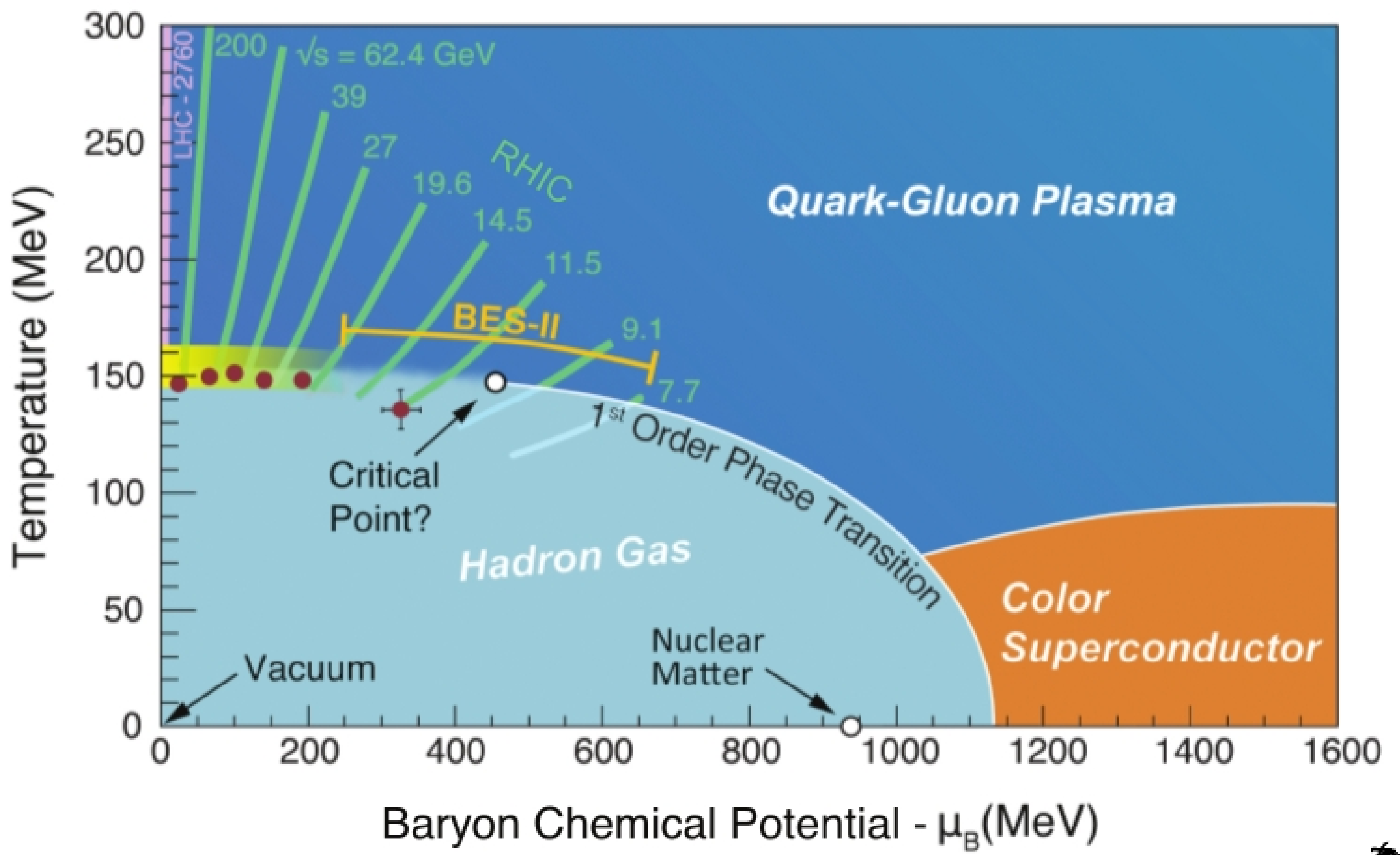


B. Friman, FK, K. Redlich V. Skokov,
Eur.Phys.J.C 71 (2011) 1694

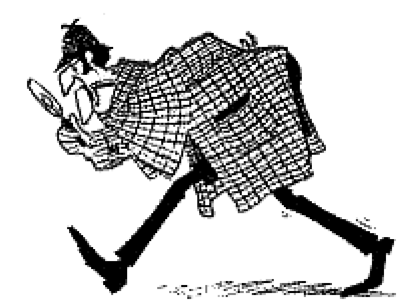


HotQCD preliminary

Exploring the phase diagram of strong-interaction matter with Taylor expansion of the QCD partition function



Where is the critical point?



QCD equation of state,critical behavior and the CEP
– Taylor expansion –

Taylor expansion of the **QCD** pressure: $\frac{P}{T^4} = \frac{1}{VT^3} \ln Z(T, V, \mu_B, \mu_Q, \mu_S)$

$$\boxed{\frac{P}{T^4} = \sum_{i,j,k=0}^{\infty} \frac{1}{i!j!k!} \chi_{ijk}^{BQS}(T) \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k}$$

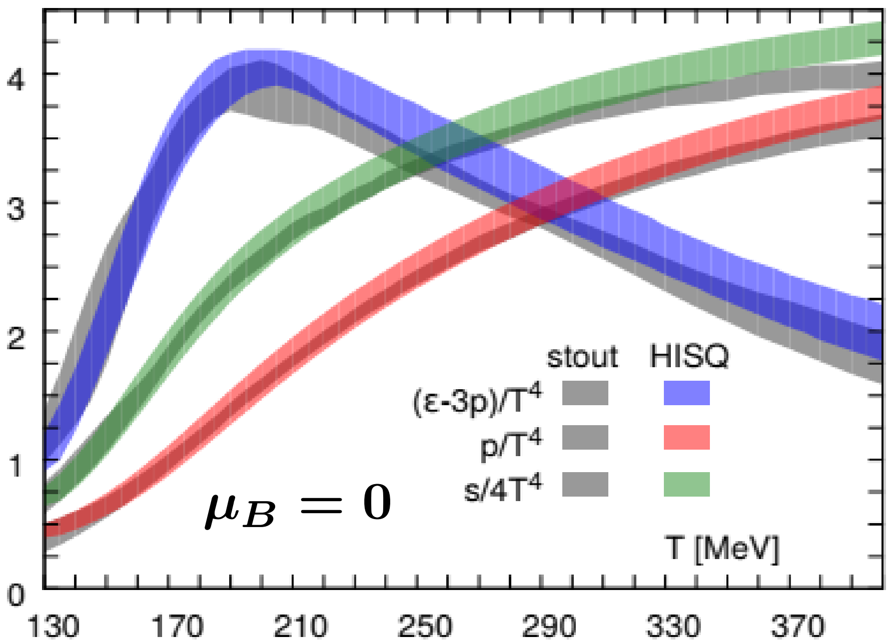
cumulants of net-charge fluctuations and correlations:

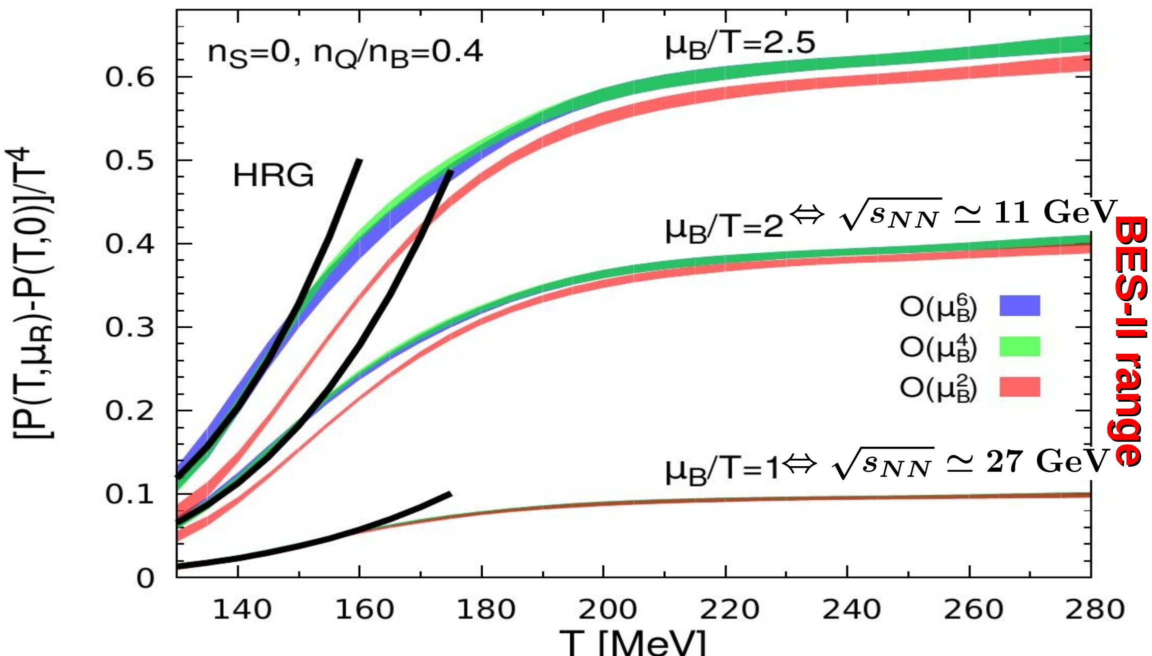
$$\chi_{ijk}^{BQS} = \left. \frac{\partial^{i+j+k} P/T^4}{\partial \hat{\mu}_B^i \partial \hat{\mu}_Q^j \partial \hat{\mu}_S^k} \right|_{\mu_{B,Q,S}=0} \quad , \quad \hat{\mu}_X \equiv \frac{\mu_X}{T}$$

Equation of state of (2+1)-flavor QCD: $\mu_B/T > 0$

$$\frac{\Delta(T,\mu_B)}{T^4} = \frac{P(T,\mu_B)-P(T,0)}{T^4} = \frac{\chi_2^B}{2}\left(\frac{\mu_B}{T}\right)^2 + \frac{\chi_4^B}{24}\left(\frac{\mu_B}{T}\right)^4 + \frac{\chi_6^B}{720}\left(\frac{\mu_B}{T}\right)^6 + \dots$$

(10-30)% contribution to total pressure at $\mu_B/T = 2$

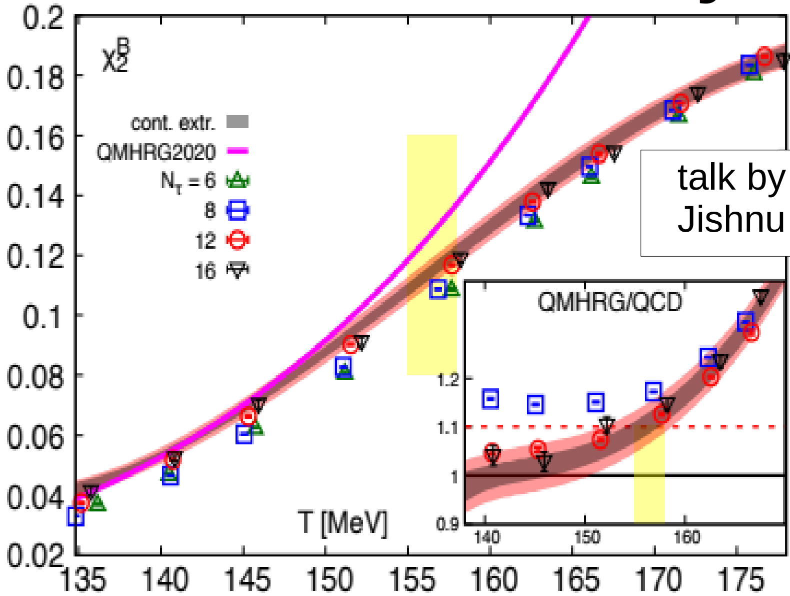




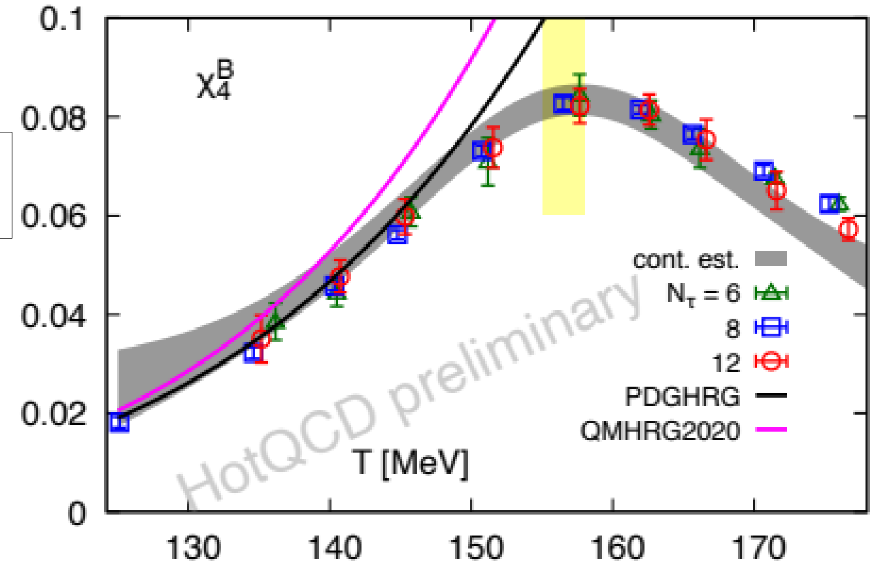
⇨ The EoS/pressure is well controlled for $\mu_B/T \le 2$ or equivalently $\sqrt{s_{NN}} \ge 11$ GeV

⇨ convergence of expansions for higher order derivatives increasingly worse

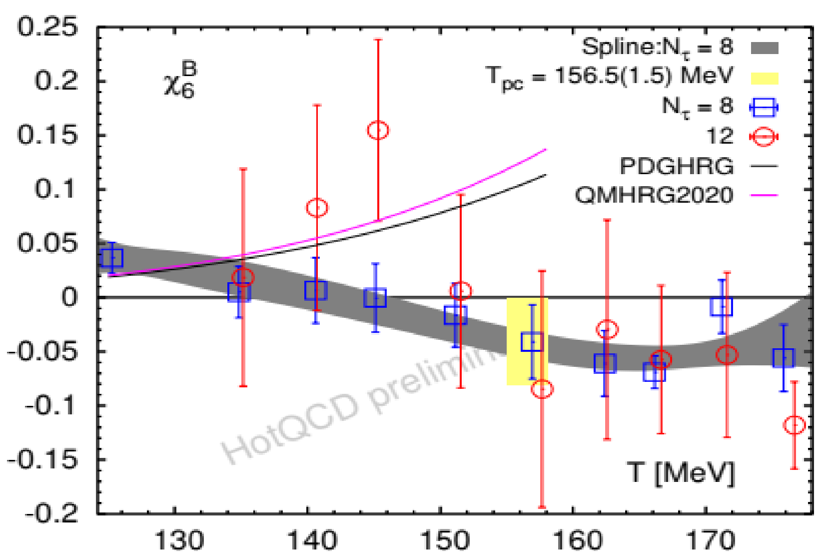
Up to 8th order cumulants are used frequently
– Taylor expansion –

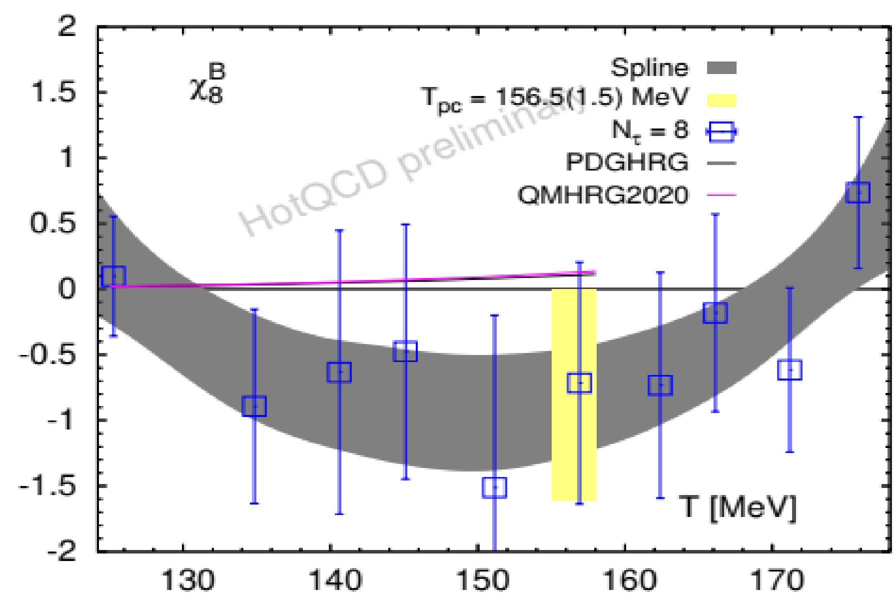


talk by
Jishnu Goswami



D. Bollweg et al. (HotQCD), arXiv:2107.10011



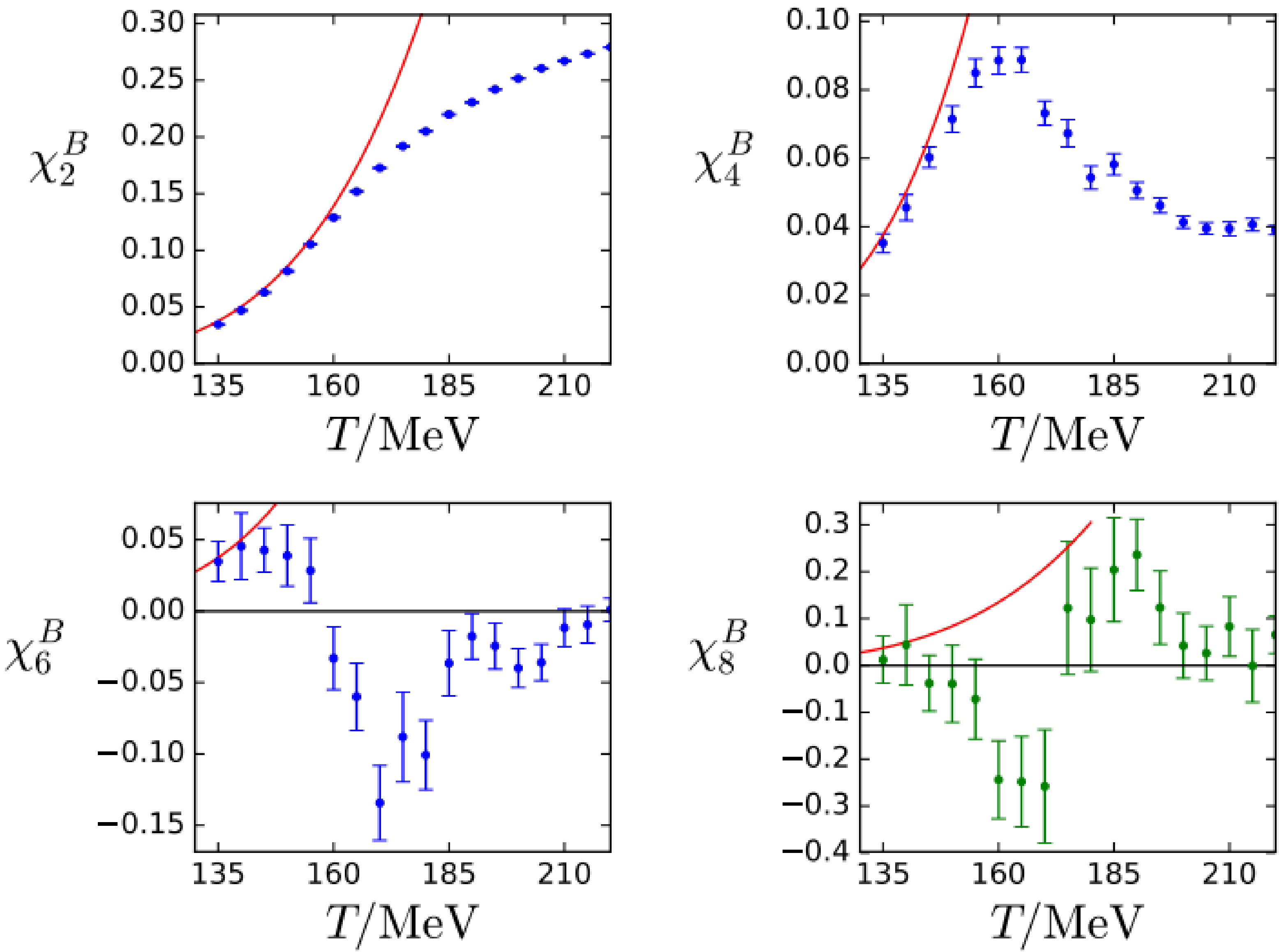


control basic features of skewness up to hyper-kurtosis ratios

A. Bazavov et al. (HotQCD), Phys. Rev. D 101 (2020) 074502, arXiv:2001.08530

Up to 8th order cumulants are used frequently
– imag. chem. pot. extrapolations –

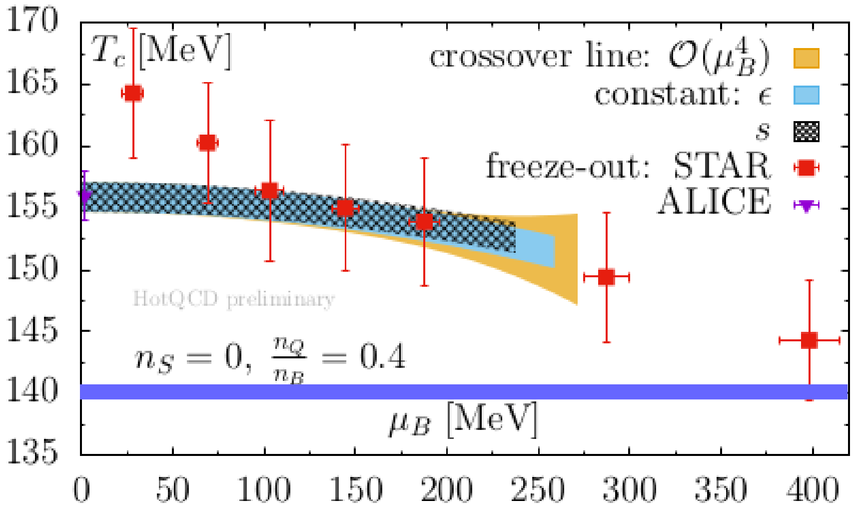
$48^3 \times 12$



note: χ_8^B is a constrained fit parameter

S. Borsanyi et al. , JHEP 10 (2018) 205, arXiv:1805.04445

Critical behavior and higher order cumulants



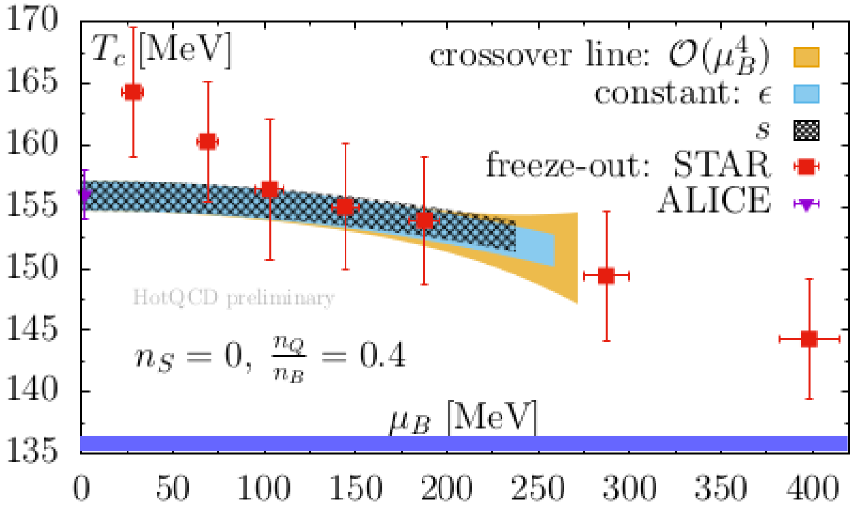
– many 8th order cumulants turn negative for

$$T^- \gtrsim (140 - 145)\ \text{MeV}$$

– higher order cumulants will continue to "oscillate" at even lower T, if convergence of Taylor series is limited by a complex zero

$$T_{CEP} < 140\text{MeV}\ ,\ \mu_B^{CEP} > 400\text{MeV}$$

Critical behavior and higher order cumulants



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$$T_{CEP} < 140\mathrm{MeV}\ ,\ \mu_B^{CEP} > 400\mathrm{MeV}$$

Lee-Yang edge zeroes uses L-QCD input

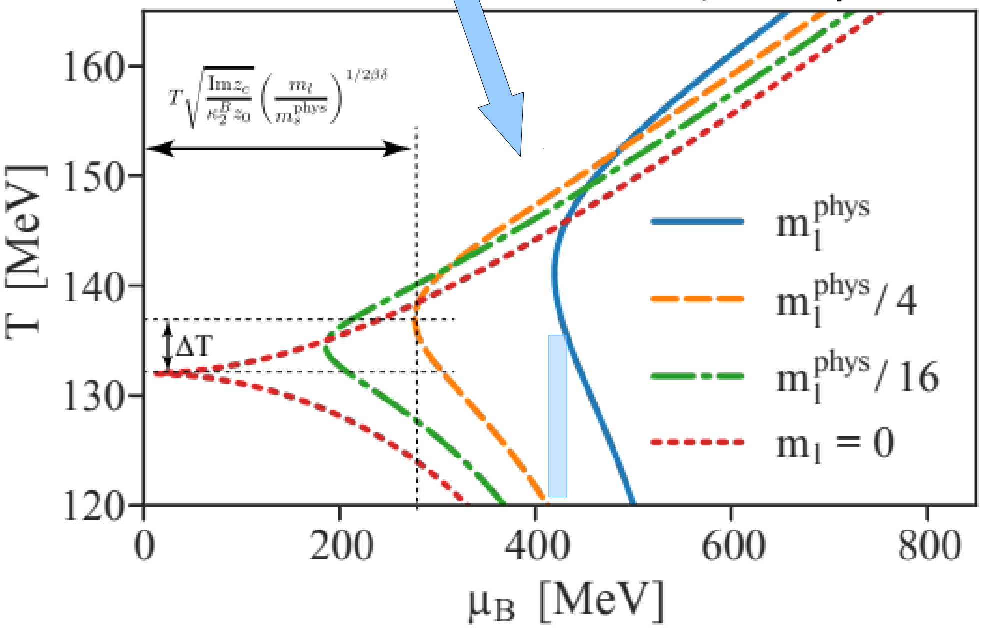
Exploit analytic structure of scaling functions:

M. A. Stephanov, hep-lat/0603014

– using input on non-universal parameters (T_c, κ_2, z_0) allows to estimate the radius of convergence deduced from universal properties of O(4) scaling function

S. Mukherjee, V. Skokov, arXiv: 1909.04639

$$f_G(z) \sim (z - z_c)^\sigma \quad , z_c \text{ complex}$$



➡ **Taylor expansions will not allow to reach the CEP, if $\mu_B^{CEP} > 400$ MeV**

➡ **Taylor series needs to be resummed**

Resumming Taylor series

– conformal mappings — V. Skokov, K. Morita, B. Friman, arXiv:1008.4549
M. Giordano et al., arXiv:2004.10800
G. Basar, arXiv:2105.08080

– partial resummation — S. Modal, S. Mukherjee, P. Hegde , arXiv:2106.03165

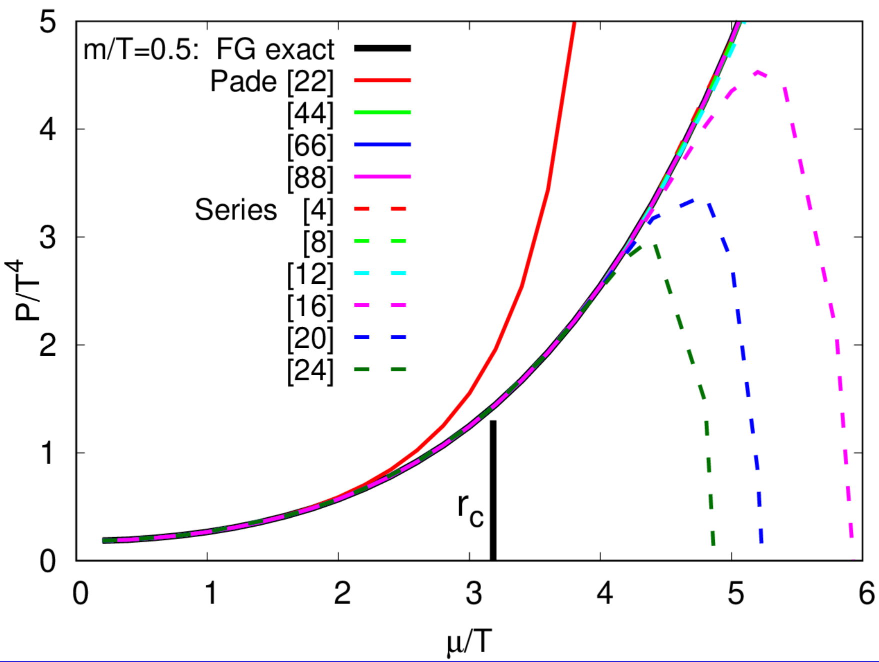
– **Pade resummation** — F. Karsch, B.-J. Schaefer, M. Wagner, J. Wambach , Phys. Lett. B 698(2011) 256, arXiv:1009.5211

a simple example:

– Taylor series for a relativistic Fermi gas as function of chemical potential
– radius of convergence controlled by an imaginary zero at $\mu_c/T = i\pi$
– series expansions in real μ break down at μ_c

– diagonal Pades, P[nn], have no problem avoiding this singularity

– phase transitions are signaled by (stable) zeroes in Pade approximants



Resumming Taylor series

$$c_n = \chi_n^B / n!$$
$$c_{n2} = c_n / c_2$$

Pade resummation

HotQCD preliminary: thanks to Jishnu Goswami, Anirban Lahiri,...

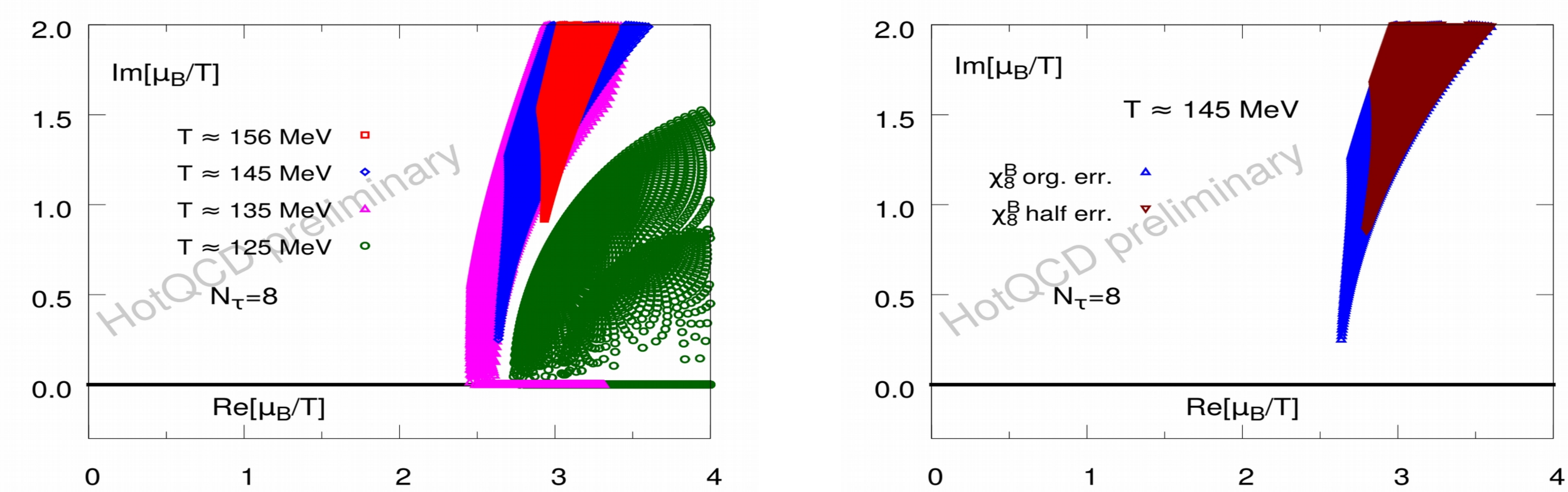
$$P(T,\mu_B)/T^4 = P(T,0)/T^4 + P_2(T,x)\ ,\ x=\mu_B/T$$

$$P_2(T,x) = c_2(T)x^2 + c_4(T)x^4 + c_6(T)x^6 + c_8(T)x^8$$

$$P[4,4](x) = c_2 \frac{(c_{42}^2 - c_{62})x^2 + \left(c_{42}^3 - 2c_{42}c_{62} + c_{82}\right)x^4}{(c_{42}^2 - c_{62}) + (c_{82} - c_{42}c_{62})x^2 + (c_{62}^2 - c_{42}c_{82})x^4}$$

– possible location of (positive) pole of the [4,4] Pade within current errors on

$$c_6 = \chi_6^B/720,\ c_8 = \chi_8^B/40320$$



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thanks to
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$$P(T,\mu_B)/T^4 = P(T,0)/T^4 + P_2(T,x)\ ,\ \ x = \mu_B/T$$

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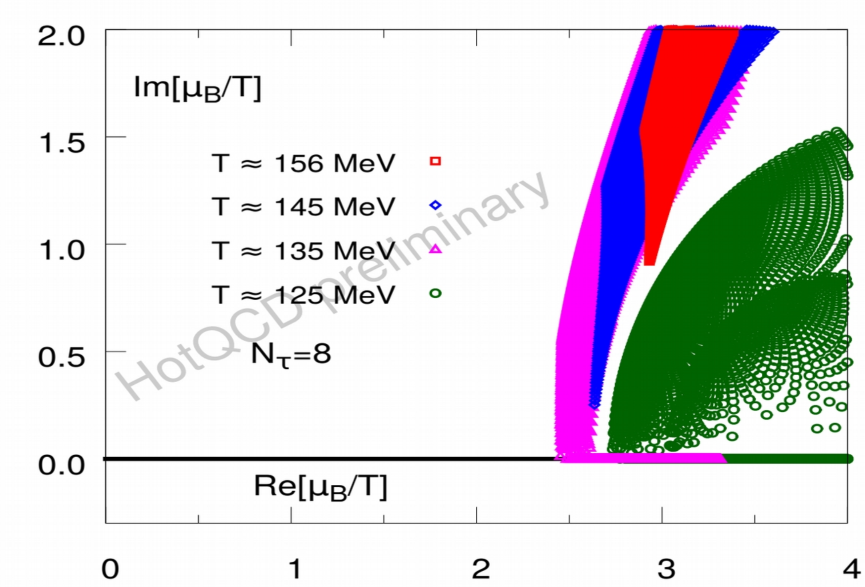


within current errors poles on the real axis (critical point) are possible only for

$$T \leq 135\mathrm{MeV}\ ,\ \ \mu_B/T > 2.5$$

higher statistics will sharpen the constraint

NB: HRG poles of [4,4] Pade at

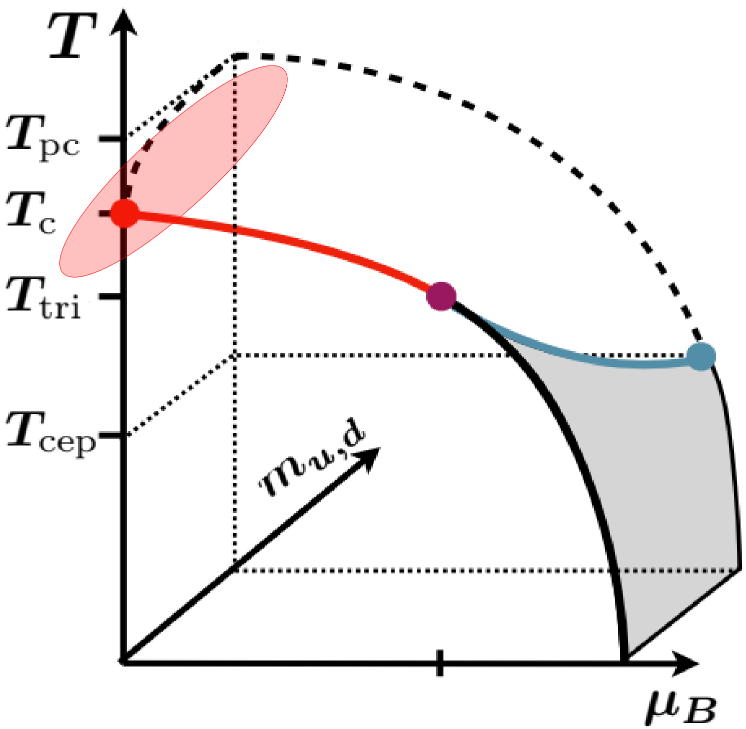
$$(\mu/T)^{poles} = \pm 5.45 \pm 2.09$$

Conclusions

- close to the chiral limit thermodynamics in the vicinity of the QCD transition(s) is controlled by a universal scaling function

singular regular

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(V,T,\vec{\mu}) = -h^{(2-\alpha)/\beta\delta} f_f(t/h^{1/\beta\delta}) - f_r(V,T,\vec{\mu})$$



What we learned so far about the CEP in QCD from lattice QCD calculations:

I) the critical temperature is below Tc=132 MeV

II) curvature of the chiral critical line suggests an even lower bound

III) the corresponding critical chemical potential is likely to be above 400 MeV

→ Taylor expansions need to be resummed in order to reach CEP