Influence of quark masses and strangeness degrees of freedom on inhomogeneous chiral phases

TECHNISCHE UNIVERSITÄT DARMSTADT

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TU Darmstadt

International School of Nuclear Physics, 42nd Course,
"QCD under extreme conditions – from heavy-ion collisions to the phase diagram"
Erice, Sicily, September 16 - 22, 2021

CRC-TR 211
Strong-interaction matter under extreme conditions

HFHF
Helmholtz Forschungsakademie Hessen für FAIR

Introduction

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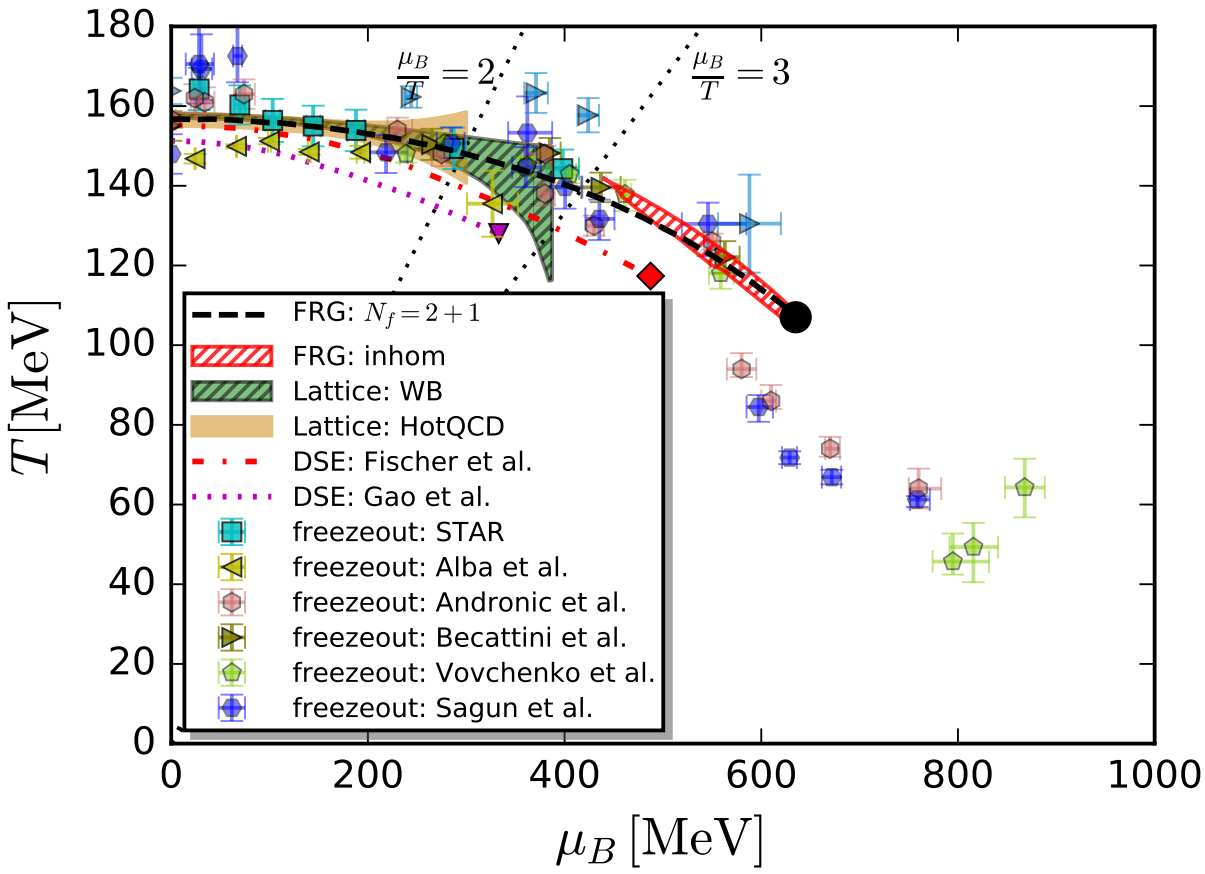
- Jan Pawlowski's talk on Friday [Fu, Pawlowski, Rennecke, PRD (2020)]:



Introduction

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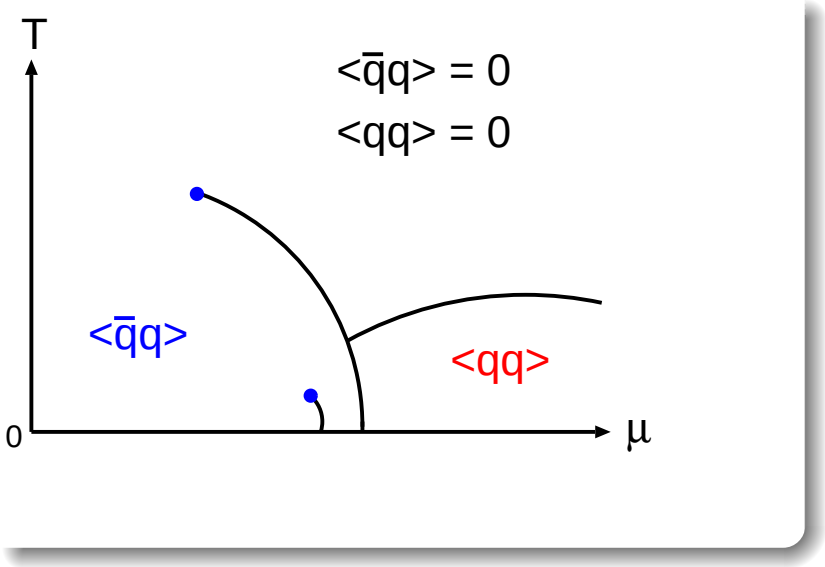
- QCD phase diagram (standard picture):



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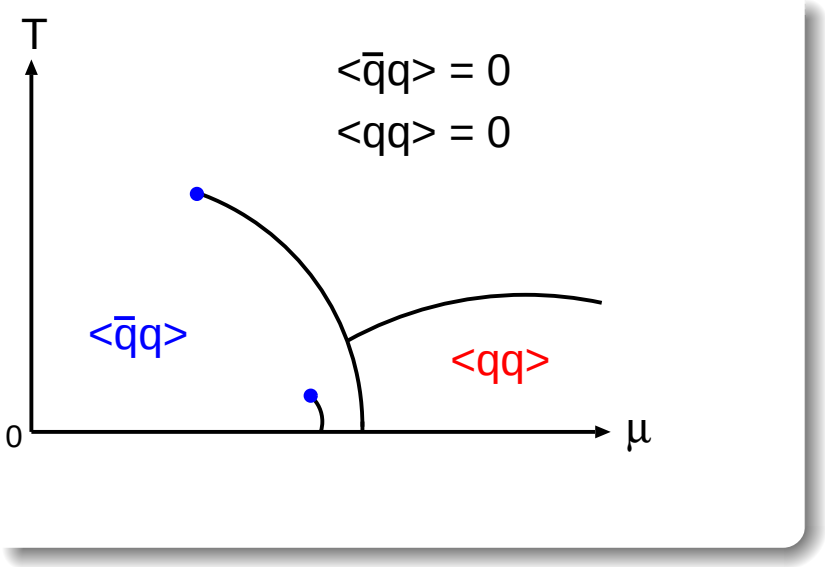

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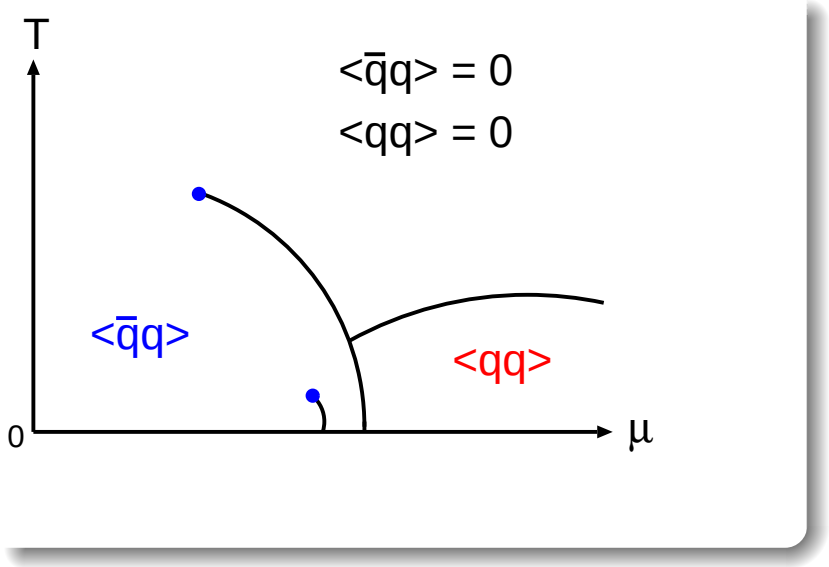
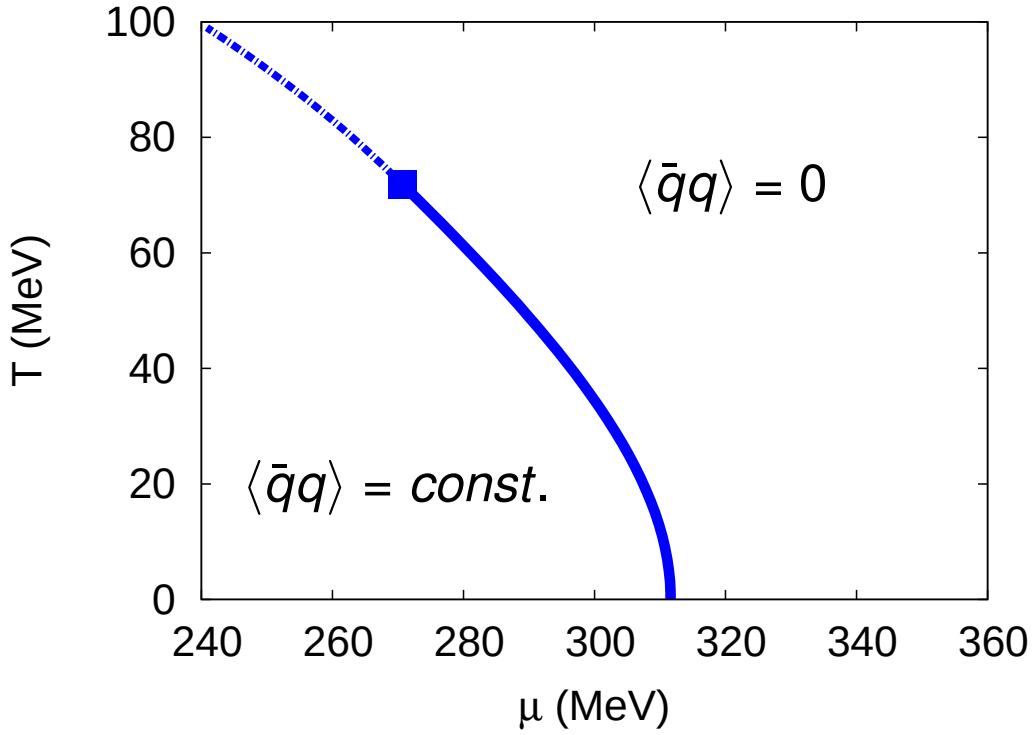


- assumption: $\langle \bar{q}q \rangle$, $\langle qq \rangle$ constant in space
- How about non-uniform phases ?

Introduction

NJL model, homogeneous phases only



[D. Nickel, PRD (2009)]

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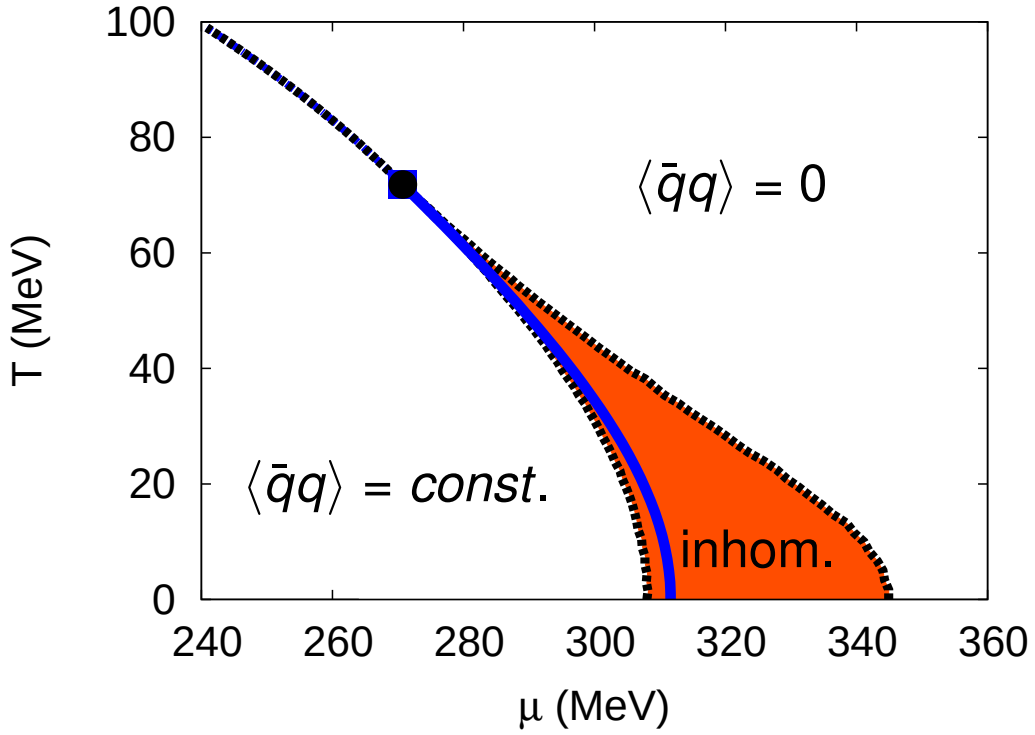
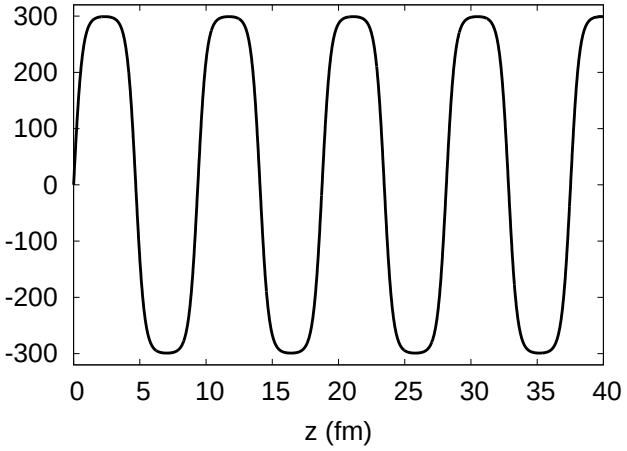
NJL model, including inhomogeneous phase



$M \propto \langle \bar{q}q \rangle$



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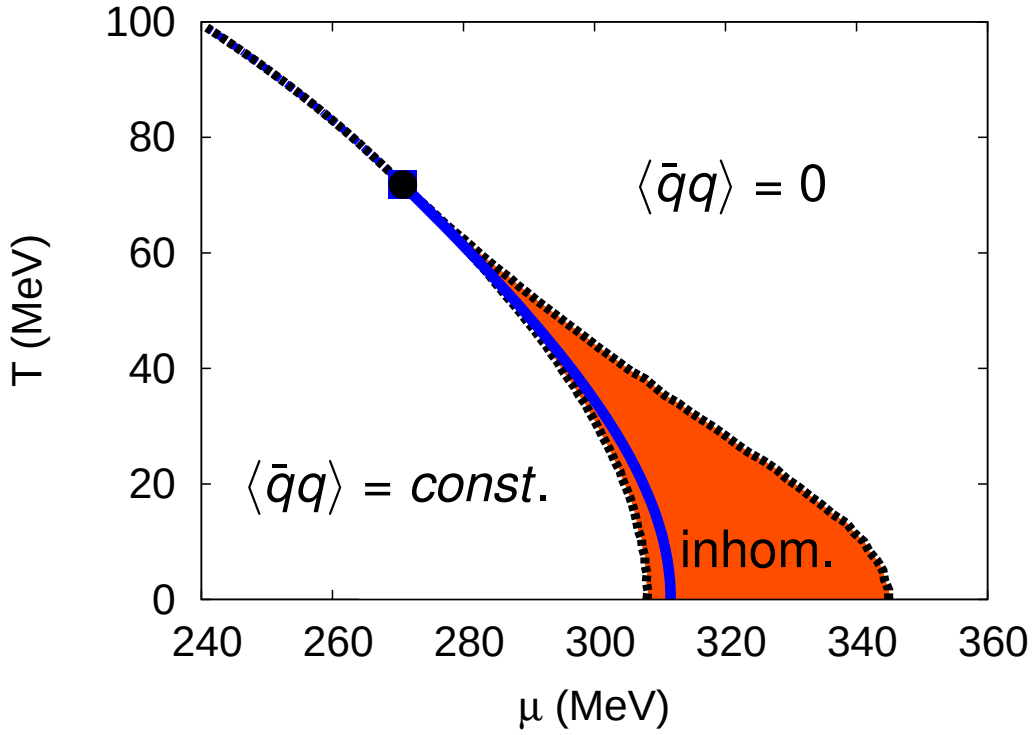

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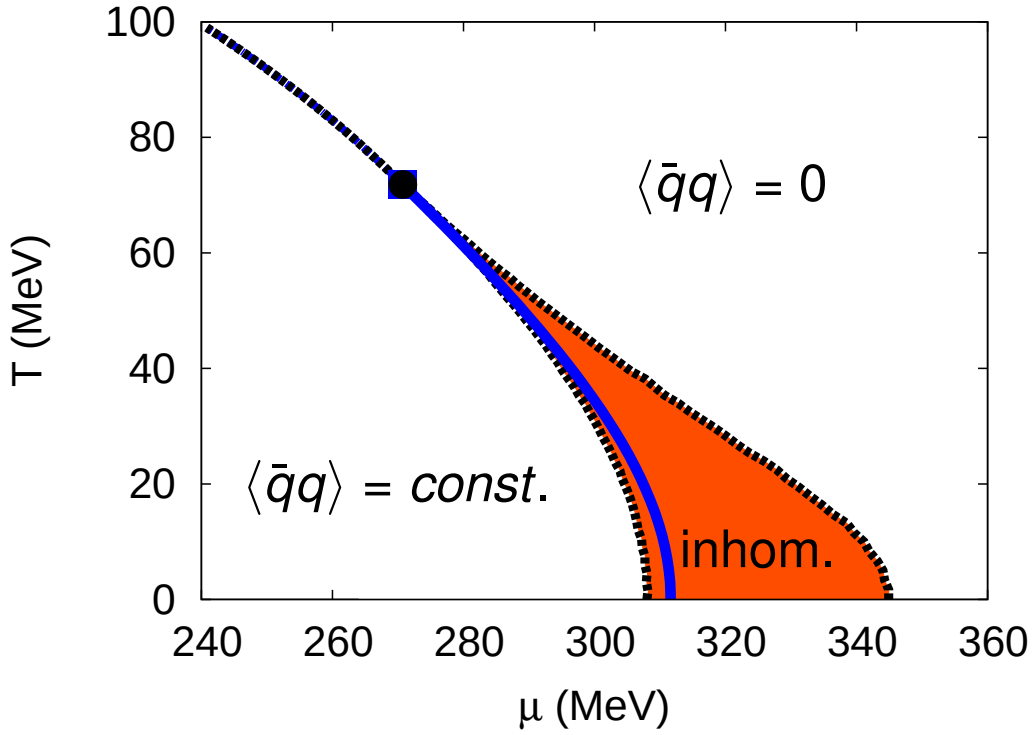

[D. Nickel, PRD (2009)]

- 1st-order phase boundary completely covered by the inhomogeneous phase!
- Critical point → Lifshitz point
  [D. Nickel, PRL (2009)]
- Inhomogeneous phase rather robust under model extensions and variations
  [MB, S. Carignano, PPNP (2015)]

Questions addressed in this talk:

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- What is the effect of explicit chiral-symmetry breaking?

  [MB, S. Carignano, PLB 791 (2019) 361;
  MB, S. Carignano, L. Kurth, EPJST 229 (2020) 3371]

- What is the influence of strange quarks?

  [S. Carignano, MB, PRD 101 (2020) 014026]

Effects of explicit chiral-symmetry breaking

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- What is the effect of going away from the chiral limit?

Effects of explicit chiral-symmetry breaking

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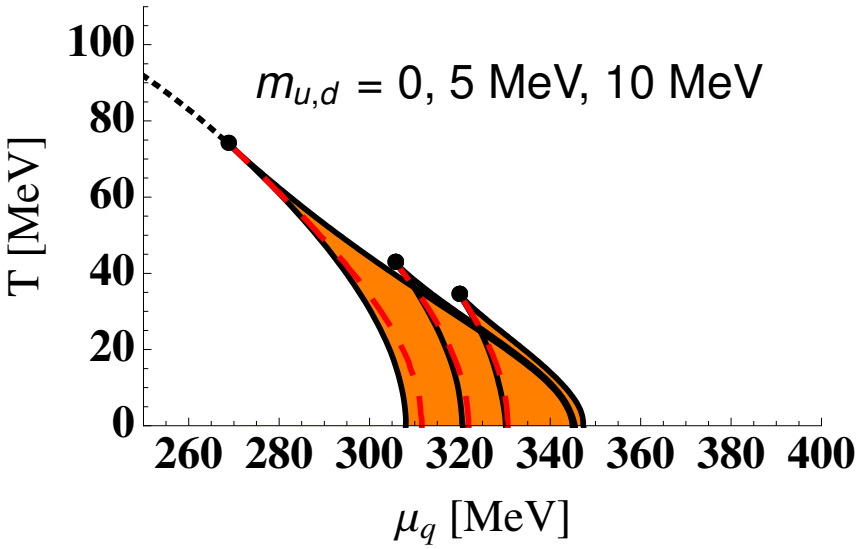

Inhomogeneous phase gets smaller but still reaches the CEP

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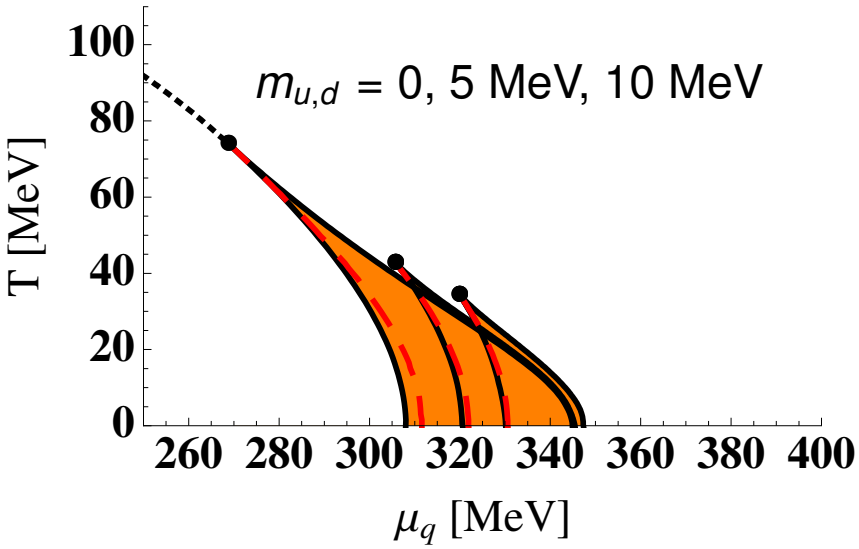

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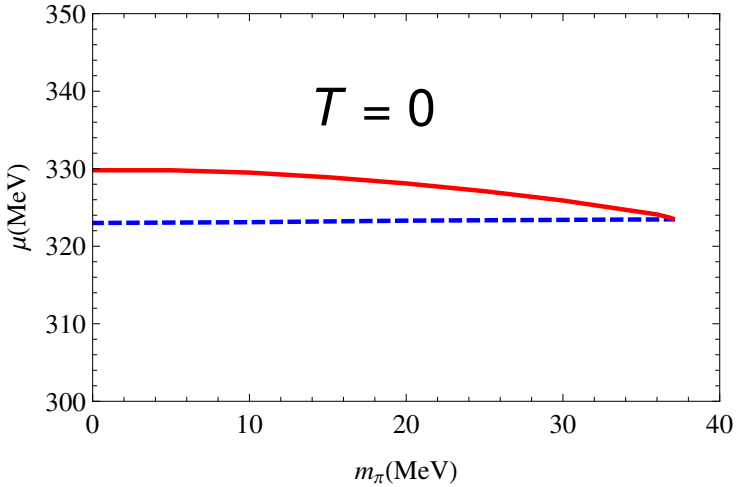
- Quark-meson model [Andersen, Kneschke, PRD (2018)]:



No inhomogeneous phase for $m_\pi > 37.1$ MeV

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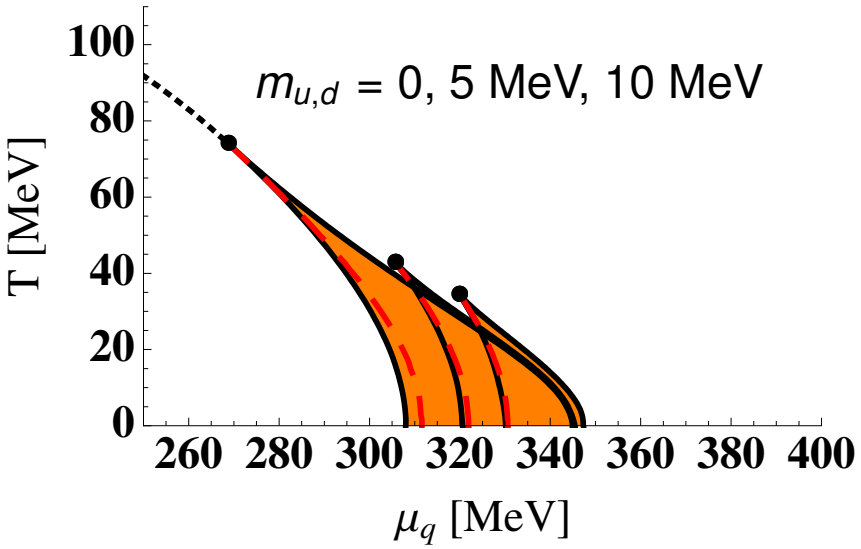

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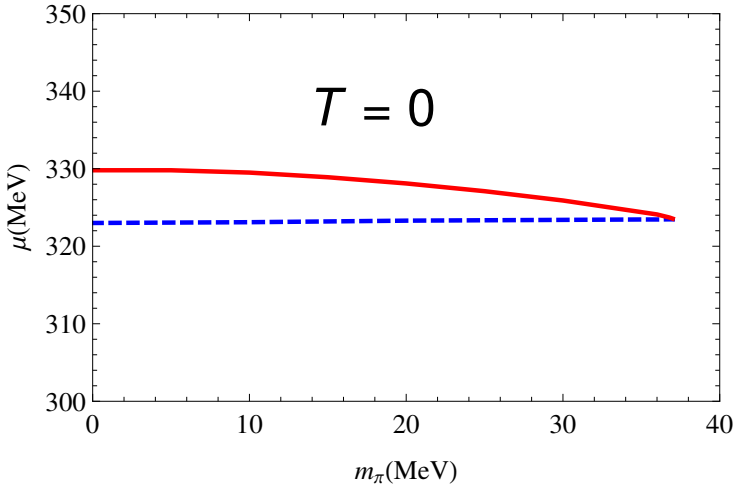

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- Here: systematic study within stablity and Ginzburg-Landau analyses

Example: Quark-Meson Model

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- Lagrangian: $\mathscr{L}_{\mathrm{QM}} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - g(\sigma + i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})\right)\psi + \mathscr{L}_{\mathrm{M}}^{\mathrm{kin}} - U(\sigma,\boldsymbol{\pi})$
  - Meson kinetic term: $\mathscr{L}_{\mathrm{M}}^{\mathrm{kin}} = \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}\right)$
  - Meson potential: $U(\sigma,\boldsymbol{\pi}) = \frac{\lambda}{4}\left(\sigma^2 + \boldsymbol{\pi}^2 - v^2\right)^2 - c\sigma$
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- Mean-field approximation:

$$\sigma(x) \to \langle\sigma(x)\rangle \equiv \sigma(\vec{x}), \qquad \pi_a(x) \to \langle\pi_a(x)\rangle \equiv \pi(\vec{x})\,\delta_{a3}$$

  - $\sigma(\vec{x})$, $\pi_(\vec{x})$ time independent classical fields
  - retain space dependence !

Mean-field thermodynamic potential

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- Grand potential per volume: $\Omega(T,\mu) = -\frac{T}{V}\log\mathcal{Z}(T,\mu)$

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- in mean-field approximation: $\Omega_{MF}(T,\mu) = \Omega_{\mathrm{q}}(T,\mu;\sigma,\pi) + \Omega_{\mathrm{mes}}(\sigma,\pi)$
  - $\Omega_{\mathrm{mes}} = \frac{1}{V}\int_V d^3x\,\left\{\frac{1}{2}\left((\boldsymbol{\nabla}\sigma(\vec{x}))^2 + (\boldsymbol{\nabla}\pi(\vec{x}))^2\right) + U(\sigma(\vec{x}),\pi(\vec{x}))\right\}$
  - $\Omega_{\mathrm{q}} = -\frac{T}{V}\mathbf{Tr}\log\frac{S^{-1}}{T}$
  - **Tr**: functional trace over Euclidean $V_4 = [0,\frac{1}{T}]\times V$, Dirac, color, and flavor
  - inverse dressed quark propagator:

    $S^{-1}(x) = i\gamma^\mu\partial_\mu + \mu\gamma^0 - g(\sigma(\vec{x}) + i\gamma_5\tau_3\pi(\vec{x}))$

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$\Rightarrow$ $\Omega_{MF} = \Omega_{MF}[\sigma(\vec{x}),\pi(\vec{x})]$

Evaluation of the functional trace + subsequent minimization extremly difficult !

Strategies

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- Stability analysis:
  - Minimize Ω_{MF} w.r.t. homogeneous mean fields $\rightarrow$ $\sigma = \bar{\sigma}$, $\pi = 0$
  - Study effect of small inhomogeneous fluctuations $\delta\sigma(\vec{x})$, $\delta\pi(\vec{x})$

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☹ instabilities w.r.t large inhomogeneous fluctuations not excluded

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- Ginzburg-Landau analysis:
  - additional expansion in small gradients $\vec{\nabla}\sigma(\vec{x})$, $\vec{\nabla}\pi(\vec{x})$
  - best suited to identify critical and Lifshitz points

Stability analysis

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- Decompose mean fields: $\sigma(\vec{x}) = \bar{\sigma} + \delta\sigma(\vec{x}), \quad \pi(\vec{x}) = \delta\pi(\vec{x})$

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- Expand thermodynamic potential in orders of $\delta\sigma$, $\delta\pi$: $\Omega_{\text{MF}} = \sum_{n=0}^{\infty}\Omega^{(n)}$
  - $\Omega^{(0)} = -\frac{T}{V}\mathbf{Tr}\log\frac{S_0^{-1}}{T} + U(\bar{\sigma}, 0)$
  - $\Omega^{(1)} = \frac{T}{V}\mathbf{Tr}\left(S_0\hat{\Sigma}\right) + \left[\lambda(\bar{\sigma}^2 - v^2)\bar{\sigma} - c\right]\frac{1}{V}\int_V d^3x\,\delta\sigma(\vec{x})$
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Evaluation of the functional traces straightforward in momentum space!

- Fourier transforms: $\delta\sigma(\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{x}}\, \delta\sigma(\vec{q}), \quad \delta\pi(\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{x}}\, \delta\pi(\vec{q})$

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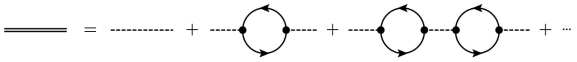
  - $D_{\sigma,\pi}(0,\vec{q})$ = sigma / pion propagator (in RPA) at zero energy and 3-momentum $\vec{q}$



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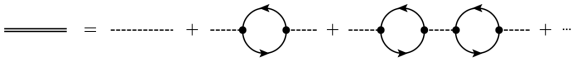

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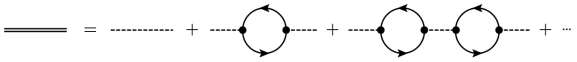

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(point particles in vacuum: $D_{\text{mes}}^{-1} = -\vec{q}^{\,2} - m_{\text{mes}}^2 < 0$)

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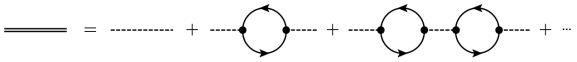

→ hom. ground state unstable if $D_\sigma^{-1}(0,\vec{q}) > 0$ or $D_\pi^{-1}(0,\vec{q}) > 0$ for some $\vec{q}$

(point particles in vacuum: $D_{\text{mes}}^{-1} = -\vec{q}^{\,2} - m_{\text{mes}}^2 < 0$)

  - 2nd-order phase boundary: $D_{\text{mes}}^{-1} = 0$ at one $|\vec{q}|$

- Fourier transforms: $\delta\sigma(\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{x}}\, \delta\sigma(\vec{q}), \quad \delta\pi(\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{x}}\, \delta\pi(\vec{q})$

→ linear term: $\Omega^{(1)} = 0$ for $\bar{\sigma}$ homogeneous minimum
($\leftrightarrow$ homogeneous gap equation)

- quadratic term: $\Omega^{(2)} = -\frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left\{ |\delta\sigma(\vec{q})|^2\, D_\sigma^{-1}(0,\vec{q}) + |\delta\pi(\vec{q})|^2\, D_\pi^{-1}(0,\vec{q}) \right\}$
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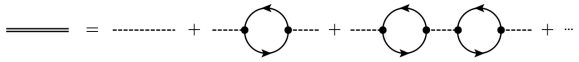

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Numerical results

[MB, S. Carignano, L. Kurth, Eur. Phys. J. ST (2020)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- Parameter fixing in vacuum:
  - chiral limit: g, λ, $v^2 \leftrightarrow f_\pi = 88$ MeV, $\bar{M} = 300$ MeV, $m_\sigma = 600$ MeV
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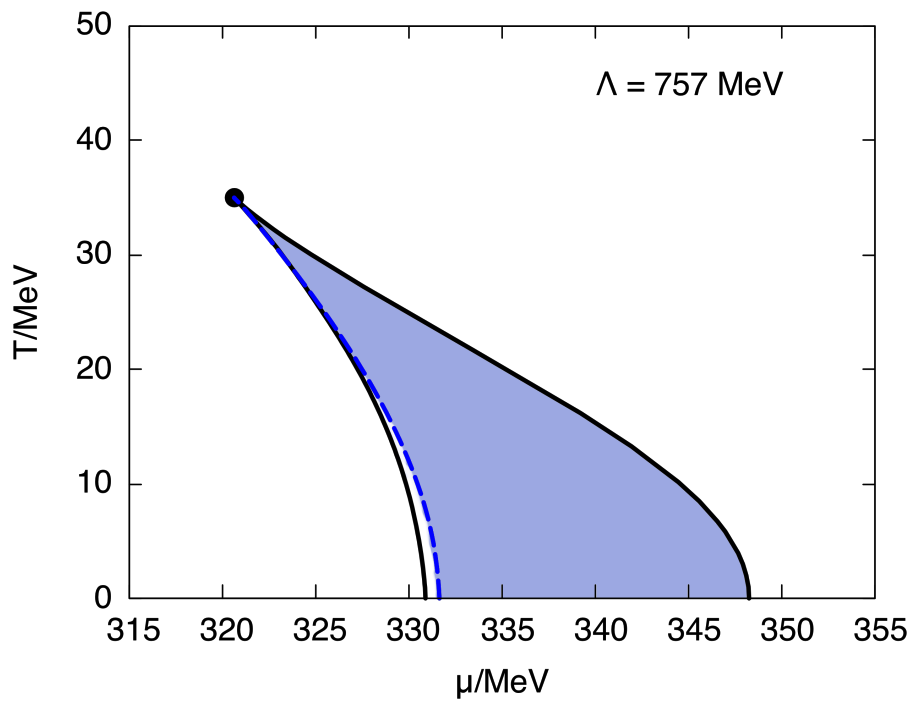

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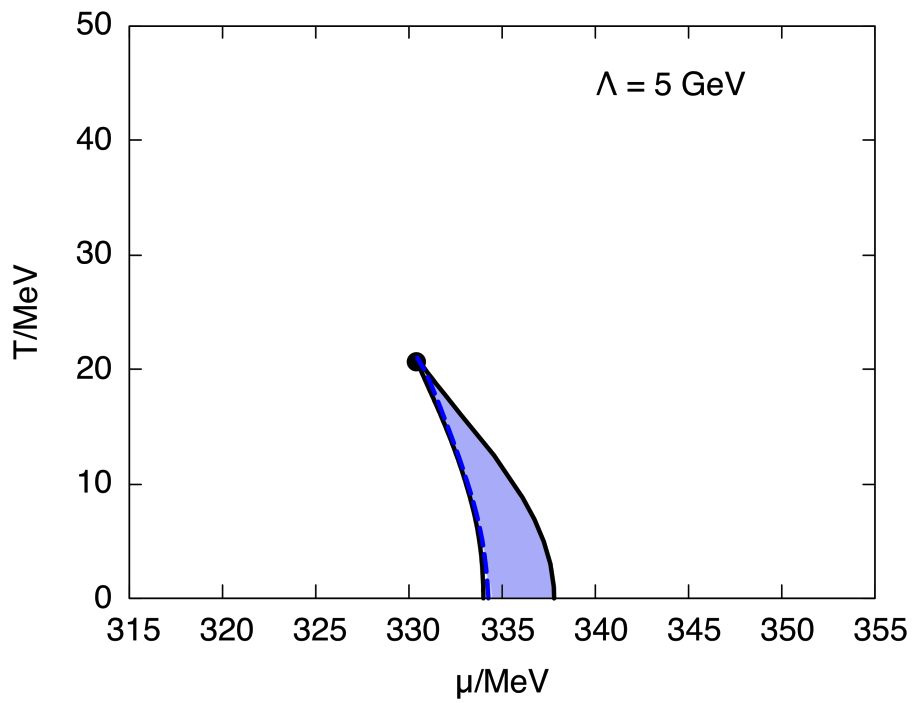

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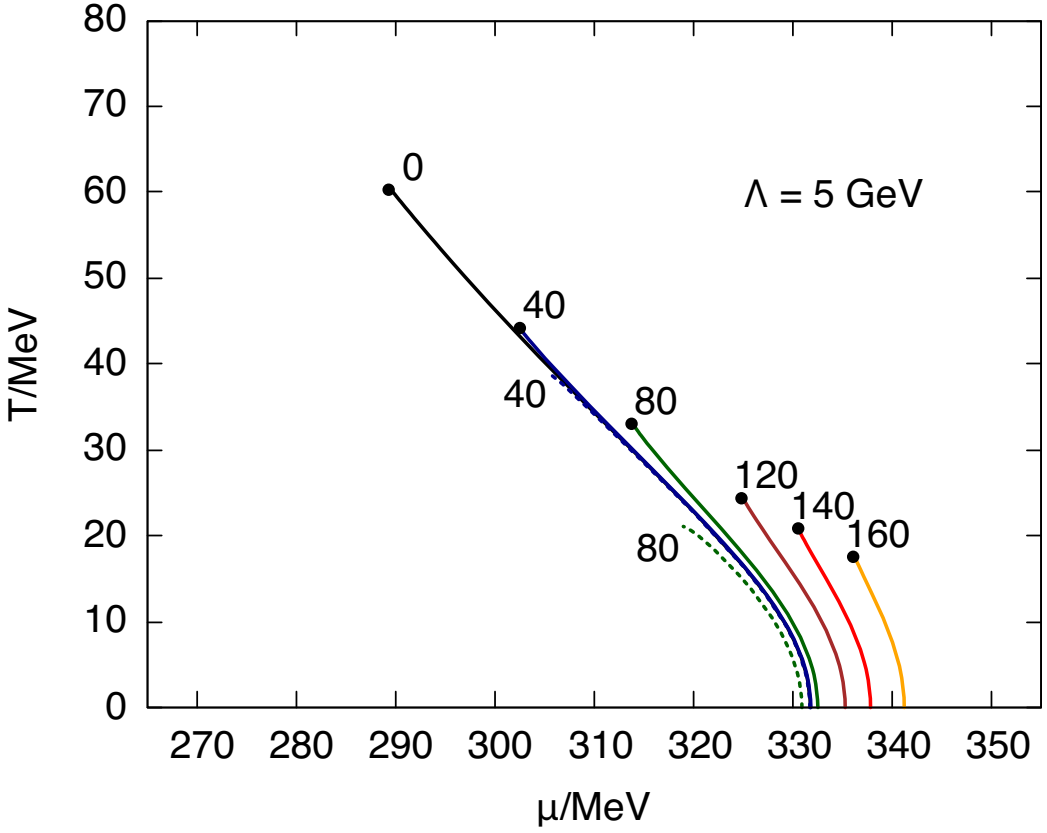

- solid lines: sigma instabilities
- dashed lines: pion instabilities

→ relevant fluctuations: $\delta\sigma$

Ginzburg-Landau analysis

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- General idea: expand Ω in powers of the order parameters and their gradients

Ginzburg-Landau analysis

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- general case: odd powers allowed, gap eq.: $\alpha_1 = 0 \rightarrow \bar{M} = \bar{M}(T, \mu)$
  - critical endpoint (CEP): $\alpha_2 = \alpha_3 = 0$
  - pseudo Lifshitz point (PLP): $\alpha_2 = \alpha_{4,b} = 0$

GL results for critical points and Lifshitz points

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	chiral limit	explicitly broken
NJL model		
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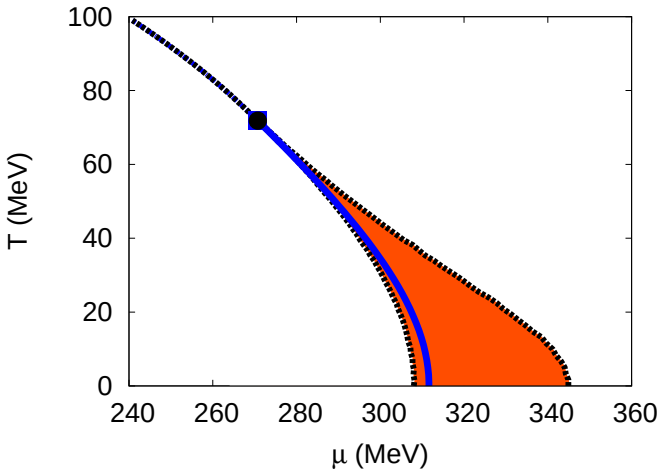

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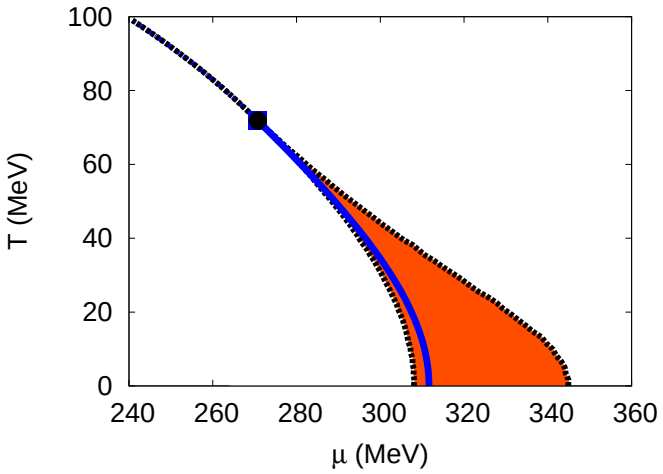

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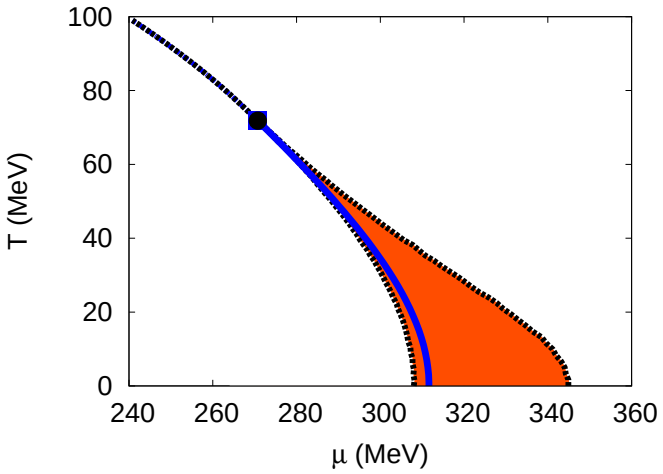

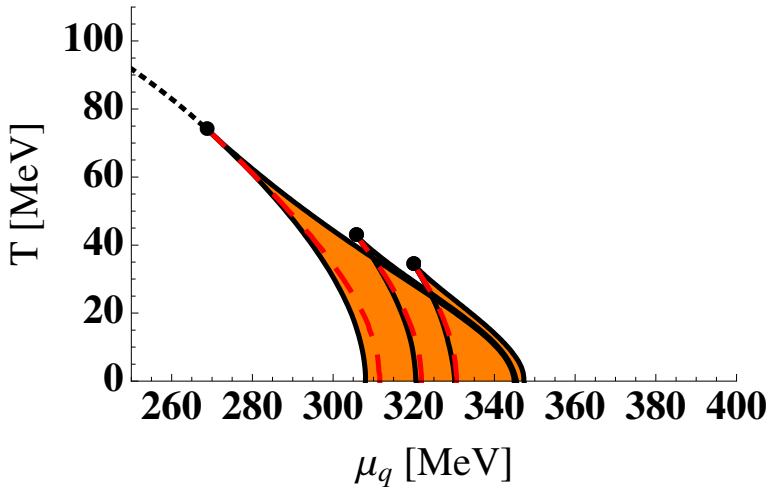

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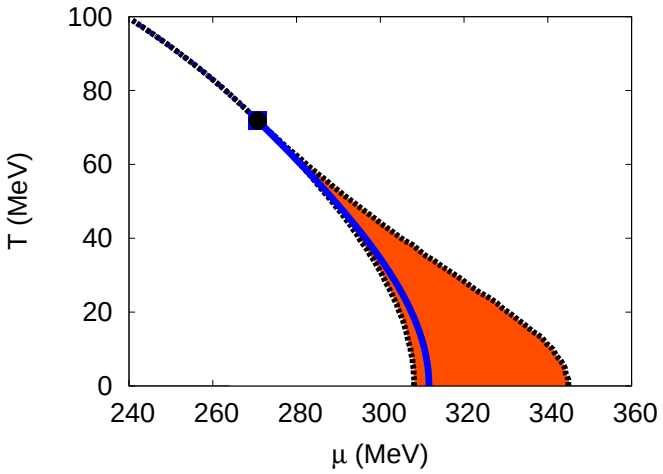

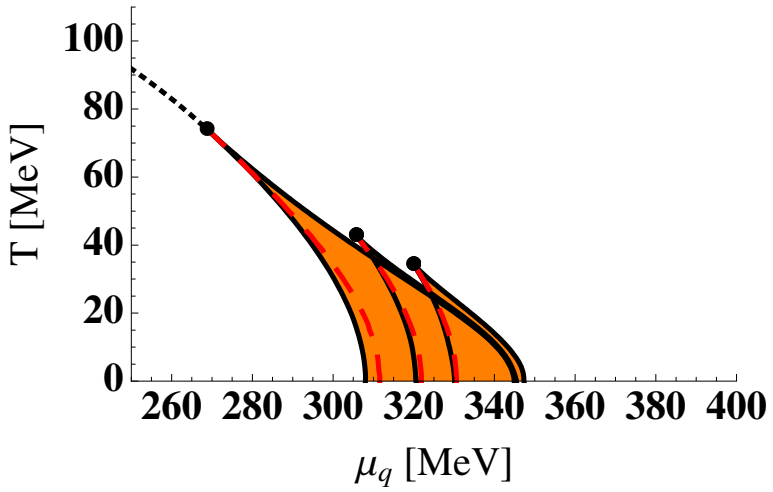

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→ Model predictions of an inhomogeneous phase should be taken as seriously as those of a CEP!

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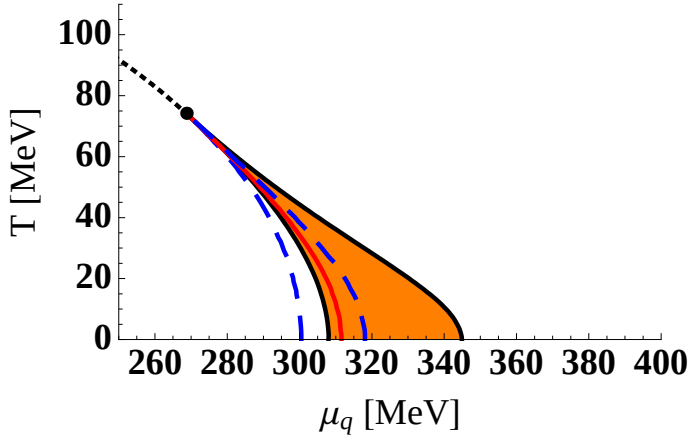

[D. Nickel, PRD (2009)]

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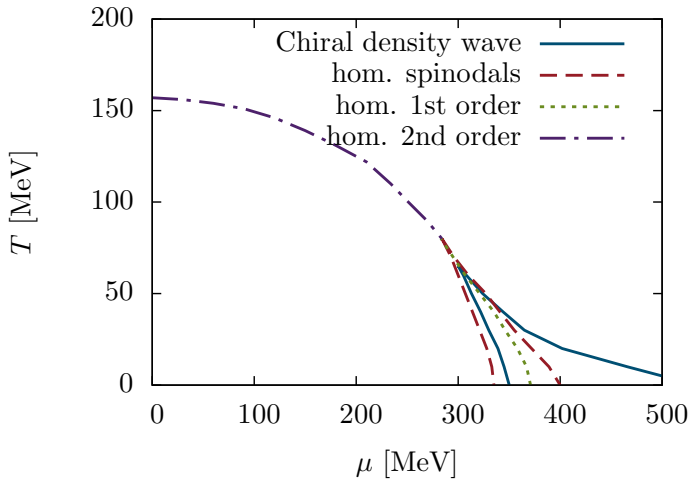

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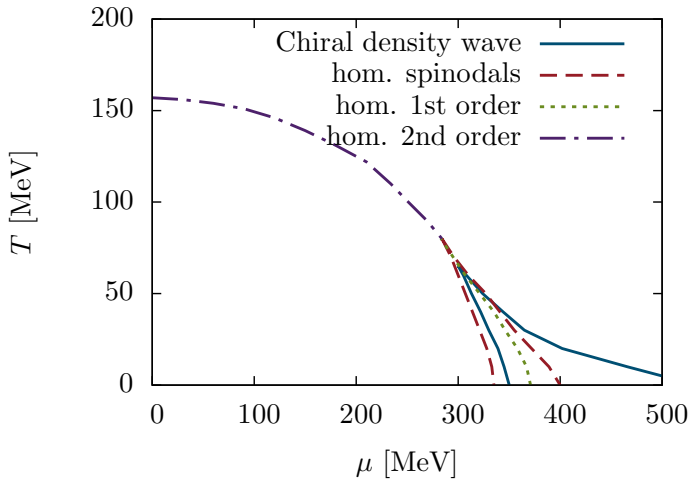

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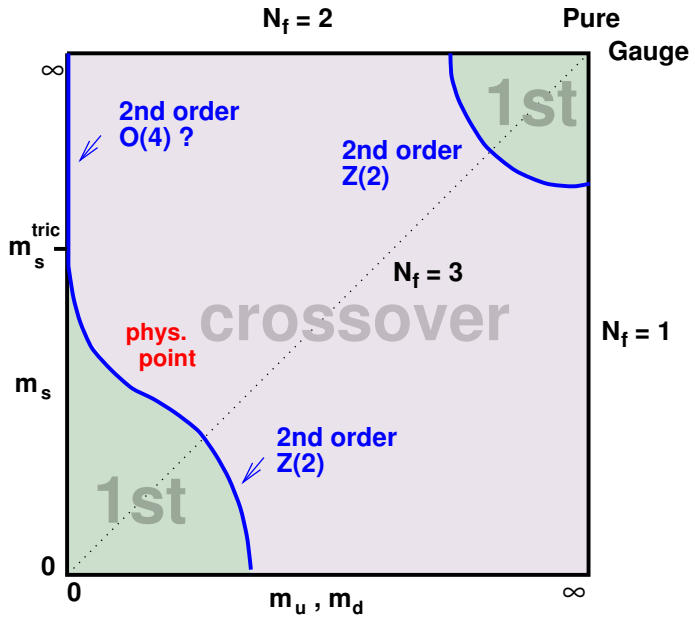

[from de Forcrand et al., POSLAT 2007]

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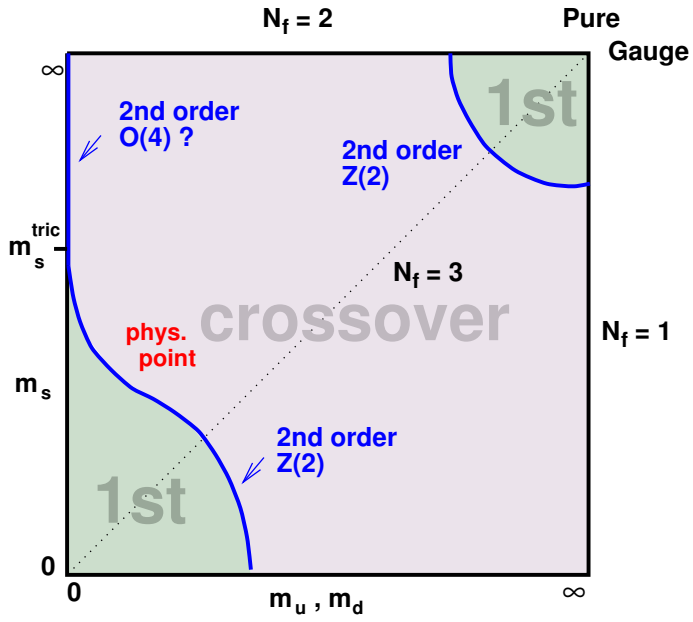

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$\rightarrow$ Chance to study the inhomogeneous phase on the lattice!

Ginzburg-Landau analysis in 2 + 1-flavor NJL

TECHNISCHE UNIVERSITÄT DARMSTADT

- GL potential for 2 massless and 1 massive quarks (order parameters: Δ_ℓ, Δ_s):

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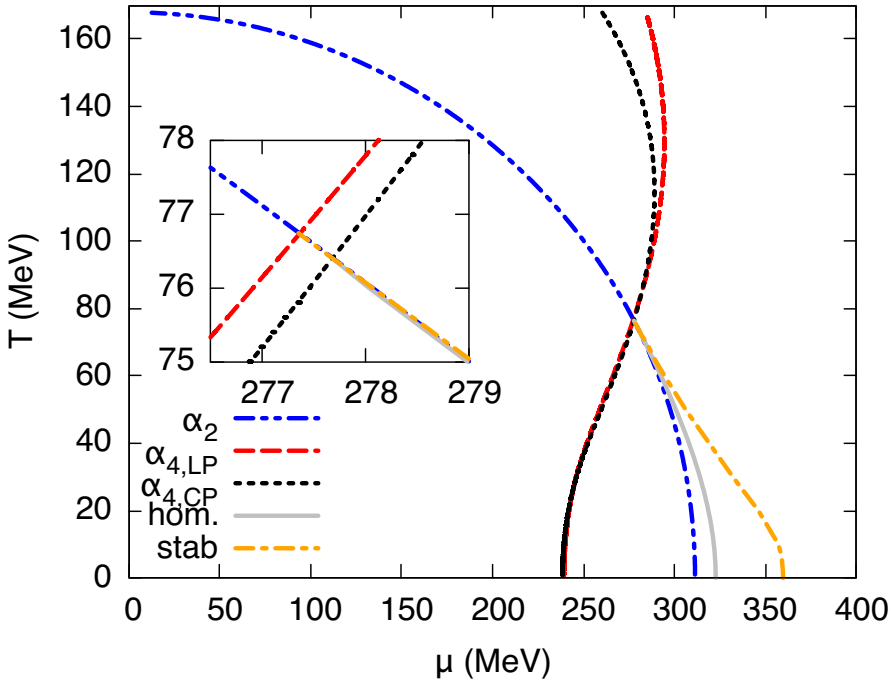
CP and LP split for $\gamma_3 \neq 0$!

Results

[S. Carignano, MB, PRD (2020)]

TECHNISCHE UNIVERSITÄT DARMSTADT

- realistic parameters (fitted to vacuum meson spectrum):



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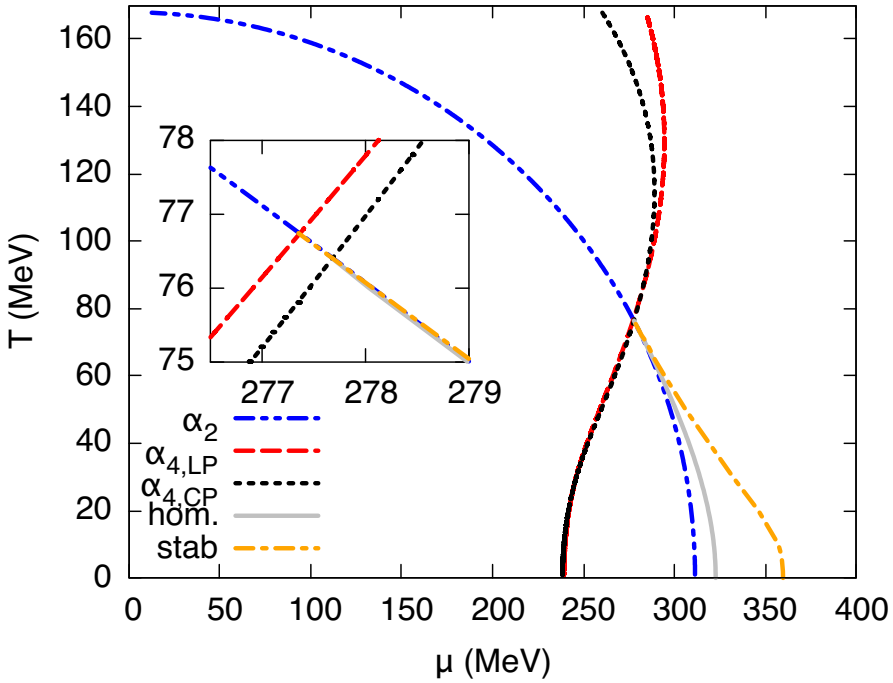

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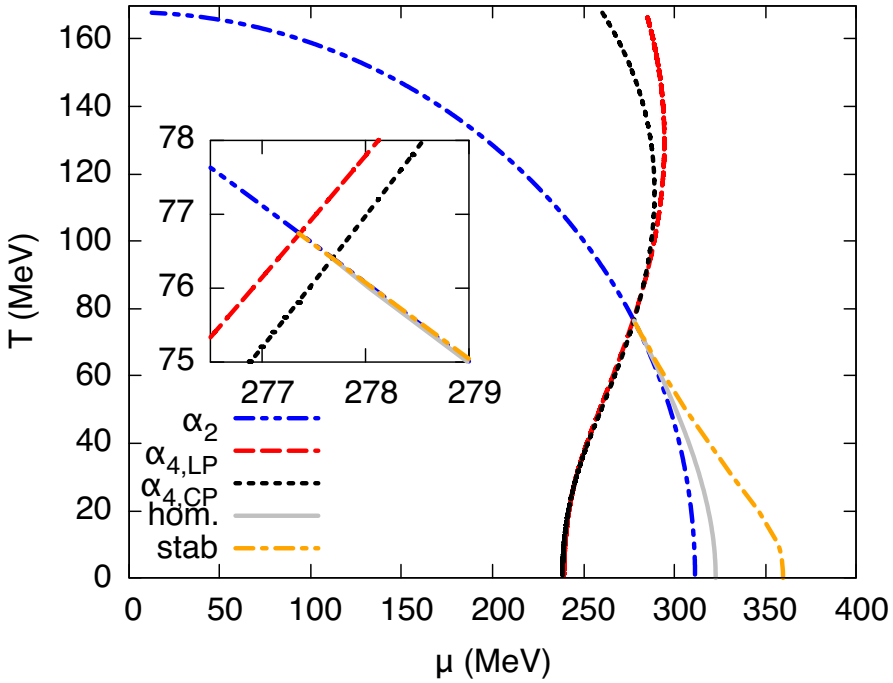
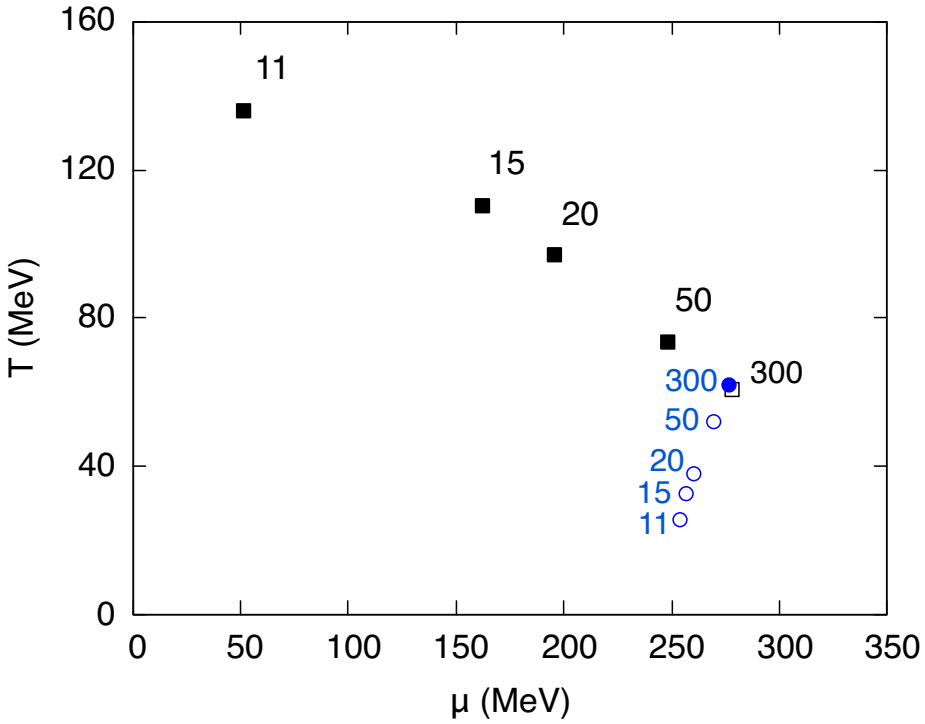


- splitting between CP and LP small
- homogeneous 1st-order phase boundary completely covered by inhom. phase

- reducing m_s:



sizeable splitting between CP and LP at small m_s

- CP $\rightarrow$ T-axis (as expected)
- LP does not follow

Conclusions

TECHNISCHE UNIVERSITÄT DARMSTADT

- Inhomogeneous chiral condensates should be considered!
- Ginzburg-Landau and stability analyses are powerful tools to identify regions of their existence.
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- Include color superconducting phases

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- Inhomogeneous chiral condensates should be considered!
- Ginzburg-Landau and stability analyses are powerful tools to identify regions of their existence.
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- Include color superconducting phases
- Stability analysis in FRG beyond mean-field approximation
- Apply to QCD using DSEs

Inhomogeneous phases beyond mean field

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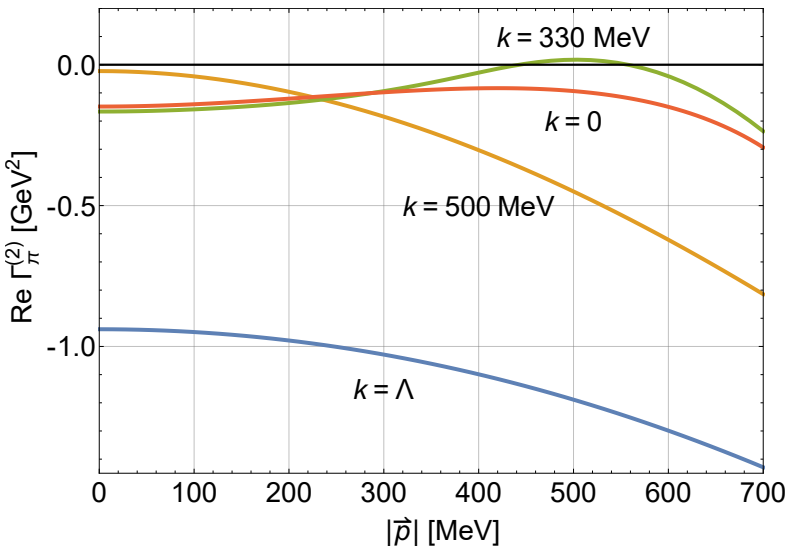
- FRG stability analysis in the QM model

  [Tripolt, Schaefer, von Smekal, Wambach, PRD (2018)]



$$\Gamma_\pi^{(2)} \equiv \Gamma_{k,\pi}^{(2)}(\omega = 0, |\vec{p}|) \sim D_{k,\pi}^{-1}(\omega = 0, |\vec{p}|)$$

- zero crossing at intermediate k
- disappears in the IR
- interpretation?
- more systematic studies needed
  (and planned within the CRC-TR 211)