Quantum Field Theory of flavor mixing

Massimo Blasone

Università di Salerno and INFN, Salerno, Italy

Summary

Motivations

• CKM quark mixing, meson mixing, massive neutrino mixing (and oscillations) play a crucial role in phenomenology;

• Theoretical interest: origin of mixing in the Standard Model;

• Bargmann superselection rule*: coherent superposition of states with different masses is not allowed in non-relativistic QM;

• Necessity of a QFT treatment: problems in defining Hilbert space for mixed particles†; oscillation formulas‡;

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*V.Bargmann, Ann. Math. (1954); D.M.Greenberger, Phys. Rev. Lett. (2001).
†C.W.Kim and A.Pevsner, *Neutrinos in Physics and Astrophysics*, (Harwood, 1993). C.Giunti, J. Phys. G (2007).
‡M.Beuthe, Phys. Rep. (2003).

Prelude

Neutrino oscillations in QM *

Pontecorvo mixing relations

$$|\nu_e\rangle = \quad \cos\theta\,|\nu_1\rangle \,+\, \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle \,+\, \cos\theta\,|\nu_2\rangle$$

– Time evolution:

$$|\nu_e(t)\rangle = \cos\theta\, e^{-iE_1 t}\,|\nu_1\rangle \,+\, \sin\theta\, e^{-iE_2 t}\,|\nu_2\rangle$$

– Flavor oscillations:

$$P_{\nu_e\to\nu_e}(t) \;=\; |\langle\nu_e|\nu_e(t)\rangle|^2 \,=\, 1 - \sin^2 2\theta\ \sin^2\left(\frac{\Delta E}{2}t\right) \,=\, 1 - P_{\nu_e\to\nu_\mu}(t)$$

– Flavor conservation:

$$|\langle\nu_e|\nu_e(t)\rangle|^2 \,+\, |\langle\nu_\mu|\nu_e(t)\rangle|^2 \,=\, 1$$

*S.M.Bilenky and B.Pontecorvo, Phys. Rep. (1978)

Mixing of neutrino fields

– Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\, \nu_1(x) + \sin\theta\, \nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\, \nu_1(x) + \cos\theta\, \nu_2(x)$$

ν_1, ν_2 are fields with definite masses.

– Mixing transformations connect the two quadratic forms:

$$\mathcal{L} = \bar{\nu}_1 \left(i \not\partial - m_1\right) \nu_1 + \bar{\nu}_2 \left(i \not\partial - m_2\right) \nu_2$$

and

$$\mathcal{L} = \bar{\nu}_e \left(i \not\partial - m_e\right) \nu_e + \bar{\nu}_\mu \left(i \not\partial - m_\mu\right) \nu_\mu - m_{e\mu} \left(\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e \right)$$

with

$$m_e = m_1 \cos^2\theta + m_2 \sin^2\theta,\ m_\mu = m_1 \sin^2\theta + m_2 \cos^2\theta,\ m_{e\mu} = (m_2 - m_1) \sin\theta \cos\theta.$$

– ν_i are free Dirac field operators:

$$\nu_i(x) = \sum_{\mathbf{k},r} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} \left[u^r_{\mathbf{k},i}(t)\, \alpha^r_{\mathbf{k},i} \;+\; v^r_{-\mathbf{k},i}(t)\, \beta^{r\dagger}_{-\mathbf{k},i} \right], \quad i = 1, 2.$$

– Anticommutation relations:

$$\{\nu_i^\alpha(x), \nu_j^{\beta\dagger}(y)\}_{t=t'} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\delta_{ij}\,; \quad \{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\} = \{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\} = \delta^3(\mathbf{k}-\mathbf{q})\delta_{rs}\delta_{ij}$$

– Orthonormality and completeness relations:

$$u^r_{\mathbf{k},i}(t) \;=\; e^{-i\omega_{k,i}t}\, u^r_{\mathbf{k},i} \quad ; \quad v^r_{\mathbf{k},i}(t) \;=\; e^{i\omega_{k,i}t}\, v^r_{\mathbf{k},i} \quad ; \quad \omega_{k,i} = \sqrt{k^2 + m_i^2}$$

$$u^{r\dagger}_{\mathbf{k},i} u^s_{\mathbf{k},i} = v^{r\dagger}_{\mathbf{k},i} v^s_{\mathbf{k},i} = \delta_{rs} \quad , \quad u^{r\dagger}_{\mathbf{k},i} v^s_{-\mathbf{k},i} = 0 \quad , \quad \sum_r (u^{r\alpha*}_{\mathbf{k},i} u^{r\beta}_{\mathbf{k},i} + v^{r\alpha*}_{-\mathbf{k},i} v^{r\beta}_{-\mathbf{k},i}) = \delta_{\alpha\beta}\ .$$

– Fock space for ν_1, ν_2:

$$\mathcal{H}_{1,2} = \left\{ \alpha^\dagger_{1,2}\ ,\ \beta^\dagger_{1,2}\ ,\ |0\rangle_{1,2} \right\}.$$

– Vacuum state $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$.

Rotation

– Pontecorvo mixing can be seen as arising by the application to the vacuum state $|0\rangle_{1,2}$ of the rotated operators:

$$
\begin{aligned}
R(\theta)^{-1}\alpha^{r\dagger}_{\mathbf{k},1}R(\theta) &= \cos\theta\,\alpha^{r\dagger}_{\mathbf{k},1} + \sin\theta\,\alpha^{r\dagger}_{\mathbf{k},2}, \\
R(\theta)^{-1}\alpha^{r\dagger}_{\mathbf{k},2}R(\theta) &= \cos\theta\,\alpha^{r\dagger}_{\mathbf{k},2} - \sin\theta\,\alpha^{r\dagger}_{\mathbf{k},1},
\end{aligned}
$$

and similar ones for $\beta^{r\dagger}_{\mathbf{k},i}$.

– The generator $R(\theta)$ is:

$$
R(\theta) \equiv \exp\left\{\theta\sum_{\mathbf{k},r}\left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{\mathbf{k},1}\beta^{r}_{\mathbf{k},2}\right) - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{\mathbf{k},2}\beta^{r}_{\mathbf{k},1}\right)\right]\right\},
$$

The above unitary operator leaves the vacuum invariant:
$R(\theta)|0\rangle_{1,2} = |0\rangle_{1,2}$

Consider the action of the rotation on the field ν_1 for example:

$$R^{-1}(\theta)\nu_1(x)R(\theta) = \cos\theta\,\nu_1(x)$$

$$+\sin\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\left(\alpha^r_{\mathbf{k},2}\,u^r_{\mathbf{k},1}(t)+\beta^{r\dagger}_{\mathbf{k},2}\,v^r_{-\mathbf{k},1}(t)\right),$$

• Problem in the last term in the r.h.s. which appears as the expansion of the field in the "wrong" basis.

Bogoliubov transformation

– We can recover the wanted expression by means of a Bogoliubov transformation:

$$\tilde{\alpha}^{r\dagger}_{\mathbf{k},i} = \cos\Theta_{\mathbf{k},i}\, \alpha^{r\dagger}_{\mathbf{k},i} - \epsilon^r\, \sin\Theta_{\mathbf{k},i}\, \beta^{r}_{-\mathbf{k},i},$$

$$\tilde{\beta}^{r\dagger}_{-\mathbf{k},i} = \cos\Theta_{\mathbf{k},i}\, \beta^{r\dagger}_{-\mathbf{k},i} + \epsilon^r\, \sin\Theta_{\mathbf{k},i}\, \alpha^{r}_{\mathbf{k},i}, \qquad i = 1, 2,$$

with $\tilde{\alpha}^{r\dagger}_{\mathbf{k},i} \equiv B_i^{-1}(\Theta_i)\, \alpha^{r\dagger}_{\mathbf{k},i}\, B_i(\Theta_i)$, etc..

– Generator

$$B_i(\Theta_i) = \exp\left\{ \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \Theta_{\mathbf{k},i}\, \epsilon^r \left[\alpha^r_{\mathbf{k},i} \beta^r_{-\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i} \alpha^{r\dagger}_{\mathbf{k},i} \right] \right\}.$$

Let us see this for the field ν_1.

$$B_2^{-1}(\Theta_2)\, R^{-1}(\theta)\, \nu_1(x)\, R(\theta)\, B_2(\Theta_2) =$$

$$= \cos\theta\, \nu_1(x) + \sin\theta \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\tilde{\alpha}^r_{\mathbf{k},2}\, u^r_{\mathbf{k},1}(t) + \tilde{\beta}^{r\dagger}_{\mathbf{k},2}\, v^r_{-\mathbf{k},1}(t) \right)$$

$$= \cos\theta\, \nu_1(x) + \sin\theta \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\alpha^r_{\mathbf{k},2}\, \tilde{u}^r_{\mathbf{k},1}(t) + \beta^{r\dagger}_{\mathbf{k},2}\, \tilde{v}^r_{-\mathbf{k},1}(t) \right) ,$$

where

$$\tilde{u}^r_{\mathbf{k},1}(t) = \cos\Theta_{\mathbf{k},2}\, u^r_{\mathbf{k},1}(t) + \epsilon^r \sin\Theta_{\mathbf{k},2}\, v^r_{-\mathbf{k},1}(t)\, ,$$

$$\tilde{v}^r_{-\mathbf{k},1}(t) = \cos\Theta_{\mathbf{k},2}\, v^r_{-\mathbf{k},1}(t) - \epsilon^r \sin\Theta_{\mathbf{k},2}\, u^r_{\mathbf{k},1}(t)\, .$$

For

$$\tilde{\Theta}_{\mathbf{k},2} = \cos^{-1}\left(u^{r\dagger}_{\mathbf{k},2}(t)u^{r}_{\mathbf{k},1}(t)\right)$$

the above Bogoliubov transformation implements the mass shift

$$\Delta m = m_2 - m_1$$

such that $\tilde{u}^{r}_{\mathbf{k},1}(t) = u^{r}_{\mathbf{k},2}(t)$ and $\tilde{v}^{r}_{-\mathbf{k},1}(t) = v^{r}_{-\mathbf{k},2}(t)$.

• The action of $B_2^{-1}(\tilde{\Theta}_2)\, R^{-1}(\theta)$ produces the desired transformation (rotation) of the field ν_1.

– Similar reasoning for ν_2, using $B_1^{-1}(\tilde{\Theta}_1)\, R^{-1}(\theta)$.

Generator of fermion mixing

Neutrino mixing in QFT

- Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\, \nu_1(x) + \sin\theta\, \nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\, \nu_1(x) + \cos\theta\, \nu_2(x)$$

can be written as†

$$\nu_e^\alpha(x) = G_\theta^{-1}(t)\, \nu_1^\alpha(x)\, G_\theta(t)$$

$$\nu_\mu^\alpha(x) = G_\theta^{-1}(t)\, \nu_2^\alpha(x)\, G_\theta(t)$$

– Mixing generator:

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

For ν_e, we get $\frac{d^2}{d\theta^2}\nu_e^\alpha = -\nu_e^\alpha$ with i.c. $\nu_e^\alpha\big|_{\theta=0} = \nu_1^\alpha$, $\frac{d}{d\theta}\nu_e^\alpha\big|_{\theta=0} = \nu_2^\alpha$.

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† M.B. and G.Vitiello, Annals Phys. (1995)

• The vacuum $|0\rangle_{1,2}$ is not invariant under the action of $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\ |0\rangle_{1,2}$$

• Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V\to\infty}\ {}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty}\ e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \ln\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right)^2} = 0$$

with

$$|V_\mathbf{k}|^2 \quad\equiv\quad \sum_{r,s} |\ v^{r\dagger}_{-\mathbf{k},1} u^s_{\mathbf{k},2}\ |^2 \neq 0 \quad for \quad m_1 \neq m_2$$

.

Quantum Field Theory vs. Quantum Mechanics

• Quantum Mechanics:

- finite $\sharp$ of degrees of freedom.

- unitary equivalence of the representations of the canonical commutation relations (von Neumann theorem).

• Quantum Field Theory:

- infinite $\sharp$ of degrees of freedom.

- ∞ many unitarily inequivalent representations of the field algebra $\Leftrightarrow$ many vacua .

- The mapping between interacting and free fields is "weak", i.e. representation dependent (LSZ formalism)*. Example: theories with spontaneous symmetry breaking.

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*F.Strocchi, *Elements of Quantum Mechanics of Infinite Systems* (World Scientific, 1985).

- The "flavor vacuum" $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state†:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[(1-\sin^2\theta\,|V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\,\cos\theta\,|V_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\,\epsilon^r \sin^2\theta\,|V_{\mathbf{k}}||U_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\,|V_{\mathbf{k}}|^2\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\,|0\rangle_{1,2}$$

- Condensation density:

$$_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; _{e,\mu}\langle 0(t)|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; \sin^2\theta\,|V_{\mathbf{k}}|^2$$

vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing).

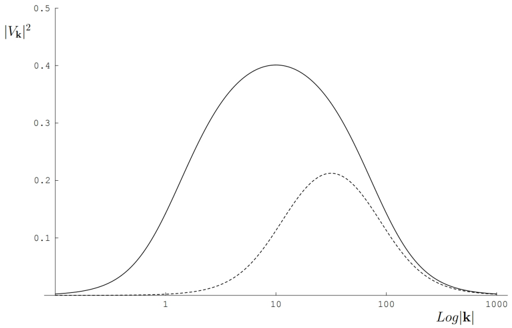
– Condensate structure as in systems with SSB (e.g. superconductors)

– Exotic condensates: mixed pairs

– Note that $|0\rangle_{e\,\mu} \neq |a\rangle_1 \otimes |b\rangle_2 \Rightarrow$ entanglement.

† A. Perelomov, *Generalized Coherent States,* (Springer V., 1986)

Condensation density for mixed fermions



Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

- $V_{\mathbf{k}} = 0$ when $m_1 = m_2$ and/or $\theta = 0$.
- Max. at $k = \sqrt{m_1 m_2}$ with $V_{max} \to \frac{1}{2}$ for $\frac{(m_2 - m_1)^2}{m_1 m_2} \to \infty$.
- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.

• Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\,\beta^r_{-\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right)$$

• Mixing transformation = Rotation + Bogoliubov transformation .

– Bogoliubov coefficients:

$$U_{\mathbf{k}}(t) \quad = \quad u^{r\dagger}_{\mathbf{k},2} u^r_{\mathbf{k},1}\; e^{i(\omega_{k,2}-\omega_{k,1})t} \quad ; \qquad V_{\mathbf{k}}(t) = \epsilon^r\; u^{r\dagger}_{\mathbf{k},1} v^r_{-\mathbf{k},2}\; e^{i(\omega_{k,2}+\omega_{k,1})t}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$$

Decomposition of mixing generator *

Mixing generator function of m_1, m_2, and θ. Try to disentangle the mass dependence from the one by the mixing angle.

Let us define:

$$R(\theta) \equiv \exp\left\{\theta \sum_{\mathbf{k},r} \left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{\mathbf{k},1}\beta^{r}_{\mathbf{k},2}\right)e^{i\psi_k} - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{\mathbf{k},2}\beta^{r}_{\mathbf{k},1}\right)e^{-i\psi_k}\right]\right\},$$

$$B_i(\Theta_i) \equiv \exp\left\{\sum_{\mathbf{k},r} \Theta_{\mathbf{k},i}\ \epsilon^r \left[\alpha^{r}_{\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}e^{i\phi_{k,i}}\right]\right\}, \qquad i = 1, 2$$

Since $[B_1, B_2] = 0$ we put

$$B(\Theta_1, \Theta_2) \equiv B_1(\Theta_1)\, B_2(\Theta_2)$$

*M.B., M.V.Gargiulo and G.Vitiello, Phys. Lett. B (2017)

• We find:

$$G_\theta \; = \; B(\Theta_1, \Theta_2) \;\; R(\theta) \;\; B^{-1}(\Theta_1, \Theta_2)$$

which is realized when the $\Theta_{\mathbf{k},i}$ are chosen as:

$$U_{\mathbf{k}} = e^{-i\psi_k} \; \cos(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})\,; \qquad V_{\mathbf{k}} = e^{\frac{(\phi_{k,1}+\phi_{k,2})}{2}} \; \sin(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})$$

The $B_i(\Theta_{\mathbf{k},i})$, $i = 1, 2$ are Bogoliubov transformations implementing a mass shift, and $R(\theta)$ is a rotation.

– Their action on the vacuum is given by:

$$|\widetilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2} \;\; = \;\; \prod_{\mathbf{k},r,i} \Big[\cos\Theta_{\mathbf{k},i} + \epsilon^r \sin\Theta_{\mathbf{k},i} \alpha^{r\dagger}_{\mathbf{k},i} \beta^{r\dagger}_{-\mathbf{k},i} \Big] |0\rangle_{1,2}$$

$$R^{-1}(\theta)|0\rangle_{1,2} \;\; = \;\; |0\rangle_{1,2} \; .$$

Bogoliubov vs Pontecorvo

Bogoliubov and Pontecorvo do not commute!

[ , ] ≠ 0

As a result, flavor vacuum gets a non-trivial term:

$$|0\rangle_{e,\mu} \equiv G_\theta^{-1}|0\rangle_{1,2} \;=\; |0\rangle_{1,2} \;+\; \left[B(m_1,m_2)\;,\; R^{-1}(\theta)\right]\; |\widetilde{0}\rangle_{1,2}$$

• Non-diagonal Bogoliubov transformation

$$|0\rangle_{e,\mu} \cong \left[\mathbb{1} + \theta\, a \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\; \tilde{V}_{\mathbf{k}} \sum_r \epsilon^r \left(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\right)\right]|0\rangle_{1,2}\,,$$

with $a \equiv \frac{(m_2-m_1)^2}{m_1 m_2}$.

Currents and charges for mixed fermions

Currents and charges for mixed fermions *

– Lagrangian in the mass basis:

$$\mathcal{L} = \bar{\nu}_m \,(i \,\partial\!\!\!/ - M_d)\, \nu_m$$

where $\nu_m^T = (\nu_1, \nu_2)$ and $M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$.

- $\mathcal{L}$ invariant under global $U(1)$ with conserved charge $Q=$ total charge.

– Consider now the $SU(2)$ transformation:

$$\nu'_m = e^{i\alpha_j \tau_j}\, \nu_m \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

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*M. B., P. Jizba and G. Vitiello, Phys. Lett. B (2001)

The associated currents are:

$$\delta\mathcal{L} = i\alpha_j \, \bar{\nu}_m \, [\tau_j, M_d] \, \nu_m = -\alpha_j \, \partial_\mu J^\mu_{m,j}$$

$$J^\mu_{m,j} = \bar{\nu}_m \, \gamma^\mu \, \tau_j \, \nu_m$$

– The charges $Q_{m,j}(t) \equiv \int d^3\mathbf{x} \, J^0_{m,j}(x)$, satisfy the $su(2)$ algebra:

$$[Q_{m,j}(t), Q_{m,k}(t)] = i \, \epsilon_{jkl} \, Q_{m,l}(t) \, .$$

– Casimir operator proportional to the total charge: $C_m = \frac{1}{2}Q$.

- $Q_{m,3}$ is conserved $\Rightarrow$ charge conserved separately for ν_1 and ν_2:

$$Q_1 = \frac{1}{2}Q + Q_{m,3} = \int d^3\mathbf{x} \, \nu_1^\dagger(x) \, \nu_1(x)$$

$$Q_2 = \frac{1}{2}Q - Q_{m,3} = \int d^3\mathbf{x} \, \nu_2^\dagger(x) \, \nu_2(x).$$

These are the flavor charges in the absence of mixing.

The currents in the flavor basis

– Lagrangian in the flavor basis:

$$\mathcal{L} = \bar{\nu}_f \left(i \not\partial - M\right) \nu_f$$

where $\nu_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$.

– Consider the $SU(2)$ transformation:

$$\nu_f' = e^{i\alpha_j \tau_j} \, \nu_f \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

– The charges $Q_{f,j} \equiv \int d^3\mathbf{x} \, J^0_{f,j}$ satisfy the $su(2)$ algebra:

$$[Q_{f,j}(t), Q_{f,k}(t)] = i \, \epsilon_{jkl} \, Q_{f,l}(t).$$

– Casimir operator proportional to the total charge $C_f = C_m = \frac{1}{2}Q$.

- $Q_{f,3}$ is not conserved $\Rightarrow$ exchange of charge between ν_e and ν_μ.

Define the flavor charges as:

$$Q_e(t) \equiv \frac{1}{2}Q + Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x)$$

$$Q_\mu(t) \equiv \frac{1}{2}Q - Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

where $Q_e(t) + Q_\mu(t) = Q$.

– We have:

$$Q_e(t) = \cos^2\theta\, Q_1 + \sin^2\theta\, Q_2 + \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

$$Q_\mu(t) = \sin^2\theta\; Q_1 + \cos^2\theta\; Q_2 - \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

In conclusion:

– In presence of mixing, neutrino flavor charges are defined as

$$Q_e(t) \equiv \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x) \quad ; \qquad Q_\mu(t) \equiv \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

– They are not conserved charges $\Rightarrow$ flavor oscillations.

– They are still (approximately) conserved in the vertex $\Rightarrow$ define flavor neutrinos as their eigenstates

- Problem: find the eigenstates of the above charges.

- Flavor charge operators are diagonal in the flavor ladder operators:

$$:: Q_\sigma(t) :: \;\equiv\; \int d^3\mathbf{x} \; :: \nu_\sigma^\dagger(x)\, \nu_\sigma(x) ::$$

$$= \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^{r}_{\mathbf{k},\sigma}(t) \,-\, \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\beta^{r}_{-\mathbf{k},\sigma}(t) \right) , \quad \sigma = e, \mu.$$

Here $::\, ...\, ::$ denotes normal ordering w.r.t. flavor vacuum:

$$:: A :: \equiv A \,-\, {}_{e,\mu}\langle 0|A|0\rangle_{e,\mu}$$

- Define flavor neutrino states with definite momentum and helicity:

$$|\nu^r_{\mathbf{k},\sigma}\rangle \;\equiv\; \alpha^{r\dagger}_{\mathbf{k},\sigma}(0)\, |0\rangle_{e,\mu}$$

– Such states are eigenstates of the flavor charges (at t=0):

$$:: Q_\sigma :: |\nu^r_{\mathbf{k},\sigma}\rangle \;=\; |\nu^r_{\mathbf{k},\sigma}\rangle$$

Neutrino oscillation formula (QFT)

– We have, for an electron neutrino state:

$$\mathcal{Q}_{\mathbf{k},\sigma}(t) \equiv \langle \nu^r_{\mathbf{k},e}| :: Q_\sigma(t) :: |\nu^r_{\mathbf{k},e}\rangle$$

$$= \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2$$

• Neutrino oscillation formula (exact result)*:

$$\mathcal{Q}_{\mathbf{k},e}(t) = 1 - |U_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} - \omega_{k,1}}{2}\, t\right) - |V_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} + \omega_{k,1}}{2}\, t\right)$$

$$\mathcal{Q}_{\mathbf{k},\mu}(t) = |U_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} - \omega_{k,1}}{2}\, t\right) + |V_{\mathbf{k}}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} + \omega_{k,1}}{2}\, t\right)$$

- For $k \gg \sqrt{m_1 m_2}$, $|U_{\mathbf{k}}|^2 \to 1$ and $|V_{\mathbf{k}}|^2 \to 0$.

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*M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999).

Lepton charge violation for Pontecorvo states[†]

– Pontecorvo states:

$$|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos\theta\,|\nu^r_{\mathbf{k},1}\rangle \;+\; \sin\theta\,|\nu^r_{\mathbf{k},2}\rangle$$

$$|\nu^r_{\mathbf{k},\mu}\rangle_P \;=\; -\sin\theta\,|\nu^r_{\mathbf{k},1}\rangle \;+\; \cos\theta\,|\nu^r_{\mathbf{k},2}\rangle\,,$$

are *not* eigenstates of the flavor charges.

$\Rightarrow$ *violation of lepton charge conservation* in the production/detection vertices, at tree level:

$${}_P\langle\nu^r_{\mathbf{k},e}|:Q_e(0):|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos^4\theta+\sin^4\theta+2|U_{\mathbf{k}}|\,\sin^2\theta\cos^2\theta \;<\; 1,$$

for any $\theta \neq 0$, $\mathbf{k} \neq 0$ and for $m_1 \neq m_2$.

[†] M. B., A. Capolupo, F. Terranova and G. Vitiello, Phys. Rev. **D** (2005)
C. C. Nishi, Phys. Rev. **D** (2008).

Other results

- Rigorous mathematical treatment for any number of flavors [*]
- Three flavor fermion mixing: CP violation[†];
- QFT spacetime dependent neutrino oscillation formula[‡];
- Boson mixing[§];
- Majorana neutrinos[¶];

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[*] K. C. Hannabuss and D. C. Latimer, J. Phys. A (2000); J. Phys. A (2003);
[†] M.B., A.Capolupo and G.Vitiello, Phys. Rev. **D** (2002)
[‡] M.B., P. Pires Pachêco and H. Wan Chan Tseung, Phys. Rev. **D**, (2003).
[§] M.B., A.Capolupo, O.Romei and G.Vitiello, Phys. Rev. **D**(2001); M.Binger and C.R.Ji. Phys. Rev. **D**(1999); C.R.Ji and Y.Mishchenko, Phys. Rev. **D**(2001); Phys. Rev. **D**(2002).
[¶] M.B. and J.Palmer, Phys. Rev. **D** (2004)

- Flavor vacuum and cosmological constant*
- Flavor vacuum induced by condensation of D-particles.†
- Geometric phase for mixed particles‡.

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*M.B., A.Capolupo, S.Capozziello, S.Carloni and G.Vitiello Phys. Lett. A (2004);
†N.E.Mavromatos and S.Sarkar, New J. Phys. (2008); N.E.Mavromatos, S.Sarkar and W.Tarantino, Phys. Rev. D (2008); Phys. Rev. D (2011).
‡M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999)

Lorentz invariance

– Canonical energy-momentum tensor for flavor fields:

$$\begin{aligned} T_{\rho\sigma} &= \bar{\nu}_e i\gamma_\rho \partial_\sigma \nu_e - \eta_{\rho\sigma}\bar{\nu}_e(i\gamma^\lambda\partial_\lambda - m_e)\nu_e \\ &+ \bar{\nu}_\mu i\gamma_\rho \partial_\sigma \nu_\mu - \eta_{\rho\sigma}\bar{\nu}_\mu(i\gamma^\lambda\partial_\lambda - m_\mu)\nu_\mu \\ &+ \eta_{\rho\sigma} m_{e\mu}(\bar{\nu}_e\nu_\mu + \bar{\nu}_\mu\nu_e) \end{aligned}$$

– Define momentum and Hamiltonian operators:

$$P^i = \int d^3\mathbf{x}\, T^{0i}; \qquad H = \int d^3\mathbf{x}\, T^{00}.$$

One finds:

$$P^i|\nu_{\mathbf{k},\sigma}\rangle = k^i|\nu_{\mathbf{k},\sigma}\rangle,$$

but

$$H|\nu_{\mathbf{k},\sigma}\rangle \neq \Omega_{\mathbf{k},\sigma}|\nu_{\mathbf{k},\sigma}\rangle.$$

- This happens because: $[H, Q_\sigma] \neq 0$.

Possible scenarios

- ν_e and ν_μ are not fundamental; the fundamental objects are ν_1 and ν_2[*];

- ν_e and ν_μ are fundamental but Poincaré invariance is broken (es.nonlinearly realized[†] as in DSR[‡]) $\Rightarrow$ modified dispersion relations;

- ν_e and ν_μ are fundamental and Poincaré invariance is recovered in the vertices.

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[*] C. Giunti and C. W. Kim, "Fundamentals of Neutrino Physics and Astrophysics," (2007)

[†] M. B., J. Magueijo, P. Pires-Pacheco, EPL (2005) ;

[‡] J. Magueijo, L. Smolin, Phys. Rev. D (2003);

Flavor mixing as a non-abelian gauge theory*

Let us return to the Lagrangian:

$$\mathcal{L} \; = \; \bar{\nu}_e \, (i \not{\partial} - m_e) \, \nu_e + \bar{\nu}_\mu \, (i \not{\partial} - m_\mu) \, \nu_\mu \; - \; m_{e\mu} \; (\, \bar{\nu}_e \, \nu_\mu \; + \; \bar{\nu}_\mu \, \nu_e \;) \, .$$

The field equations:

$$\begin{aligned} i\partial_0 \nu_e &= (-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m_e)\nu_e + \beta m_{e\mu}\nu_\mu \\ i\partial_0 \nu_\mu &= (-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m_\mu)\nu_\mu + \beta m_{e\mu}\nu_e . \end{aligned}$$

can be written compactly:

$$iD_0 \nu_f = (-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta M_d)\nu_f ,$$

with $\nu_f = (\nu_e, \nu_\mu)^T$, $M_d = \mathrm{diag}(m_e, m_\mu)$.

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*M. B., M. Di Mauro, G. Vitiello, Phys. Lett. B (2011)

- Non-abelian covariant derivative:

$$D_0 := \partial_0 + i\, m_{e\mu}\, \beta\, \sigma_1,$$

with $m_{e\mu} = \frac{1}{2}\, \tan 2\theta\, \delta m$ and $\delta m := m_\mu - m_e$.

- Gauge connection:

$$A_\mu \quad := \quad \frac{1}{2} A^a_\mu \sigma_a \;=\; n_\mu \delta m\, \frac{\sigma_1}{2} \in su(2),$$

with $n^\mu := (1, 0, 0, 0)^T$, so that:

$$D_\mu = \partial_\mu + i\, g\, \beta\, A_\mu.$$

We define $g \equiv \tan 2\theta$ as the coupling constant for the mixing interaction.

- The equations of motion and the Lagrangian read:

$$(i\gamma^\mu D_\mu - M_d)\nu_f = 0,$$

$$\mathcal{L} = \bar{\nu}_f (i\gamma^\mu D_\mu - M_d)\nu_f.$$

- Define a new energy-momentum tensor:

$$\widetilde{T}_{\rho\sigma} = \bar{\nu}_f i\gamma_\rho D_\sigma \nu_f - \eta_{\rho\sigma}\bar{\nu}_f(i\gamma^\lambda D_\lambda - M_d)\nu_f .$$

- Momentum and Hamiltonian operators:

$$\begin{aligned}
\widetilde{P}^i &= \int d^3\mathbf{x}\, \widetilde{T}^{0i} \\
&= i\int d^3\mathbf{x}\, \nu_e^\dagger \partial^i \nu_e + i\int d^3\mathbf{x}\, \nu_\mu^\dagger \partial^i \nu_\mu \\
&\equiv \widetilde{P}_e^i(t) + \widetilde{P}_\mu^i(t), \qquad i = 1, 2, 3;
\end{aligned}$$

$$\begin{aligned}
\widetilde{H}(t) &= \int d^3\mathbf{x}\, \widetilde{T}^{00} \\
&= \int d^3\mathbf{x}\, \nu_e^\dagger \left(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m_e\right)\nu_e + \int d^3\mathbf{x}\, \nu_\mu^\dagger \left(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m_\mu\right)\nu_\mu \\
&\equiv \widetilde{H}_e(t) + \widetilde{H}_\mu(t).
\end{aligned}$$

This Hamiltonian does *not* generate time evolution.

Flavor fields in a different mass basis

– Flavor fields can be expanded also as*

$$\nu_\sigma(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_r \left[u^r_{\mathbf{k},\sigma}(t) \widetilde{\alpha}^r_{\mathbf{k},\sigma}(t) + v^r_{-\mathbf{k},\sigma}(t) \widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(t) \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \sigma = e, \mu,$$

with $u^r_{\mathbf{k},\sigma}(t) = u^r_{\mathbf{k},\sigma} e^{-i\omega_{\mathbf{k},\sigma} t}$ and $v^r_{-\mathbf{k},\sigma}(t) = v^r_{-\mathbf{k},\sigma} e^{i\omega_{\mathbf{k},\sigma} t}$.

The spinor basis is defined by:

$$\begin{aligned} (-\alpha \cdot \mathbf{k} + m_\sigma \beta) u^r_{\mathbf{k},\sigma} &= \omega_{\mathbf{k},\sigma} u^r_{\mathbf{k},\sigma} \\ (-\alpha \cdot \mathbf{k} + m_\sigma \beta) v^r_{-\mathbf{k},\sigma} &= -\omega_{\mathbf{k},\sigma} v^r_{-\mathbf{k},\sigma}, \end{aligned}$$

where $\omega_{\mathbf{k},\sigma} = \sqrt{\mathbf{k}^2 + m^2_\sigma}$.

*K. Fujii, C. Habe, T. Yabuki Phys. Rev. D (1999);

– Operators in different bases are connected by a Bogoliubov transformation:

$$\left(\begin{array}{c} \widetilde{\alpha}^r_{\mathbf{k},\sigma}(t) \\ \widetilde{\beta}^{r\dagger}_{-\mathbf{k},\sigma}(t) \end{array} \right) = J^{-1}(t) \left(\begin{array}{c} \alpha^r_{\mathbf{k},\sigma}(t) \\ \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \end{array} \right) J_\mu(t),$$

with generator:

$$J(t) = \prod_{\mathbf{k},r} \exp\left\{ i \sum_{(\sigma,j)} \xi^{\mathbf{k}}_{\sigma,j} \left[\alpha^{r\dagger}_{\mathbf{k},\sigma}(t) \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) + \beta^{r}_{-\mathbf{k},\sigma}(t) \alpha^{r}_{\mathbf{k},\sigma}(t) \right] \right\},$$

where $(\sigma, j) = (e, 1), (\mu, 2)$, $\xi^{\mathbf{k}}_{\sigma,j} = (\chi_\sigma - \chi_j)/2$ and $\chi_\sigma = \arctan(\mu_\sigma/|\mathbf{k}|)$, $\chi_j = \arctan(m_j/|\mathbf{k}|)$.

– New flavor vacuum:

$$|\widetilde{0}(t)\rangle_{e\mu} = J^{-1}(t)|0(t)\rangle_{e\mu}.$$

– Momentum and Hamiltonian operators are both diagonalized:

$$\widetilde{\mathbf{P}}_\sigma(t) \;=\; \sum_r \int d^3\mathbf{k}\; \mathbf{k} \left(\widetilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}(t)\widetilde{\alpha}^{r}_{\mathbf{k},\sigma}(t) + \widetilde{\beta}^{r\dagger}_{\mathbf{k},\sigma}(t)\widetilde{\beta}^{r}_{\mathbf{k},\sigma}(t)\right),$$

$$\widetilde{H}_\sigma(t) \;=\; \sum_r \int d^3\mathbf{k}\; \omega_{\mathbf{k},\sigma} \left(\widetilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}(t)\,\widetilde{\alpha}^{r}_{\mathbf{k},\sigma}(t) - \widetilde{\beta}^{r}_{\mathbf{k},\sigma}(t)\,\widetilde{\beta}^{r\dagger}_{\mathbf{k},\sigma}(t)\right).$$

– Flavor charges remain diagonal ($[Q_\sigma(t), J(t)] = 0$):

$$\widetilde{Q}_\sigma(t) = \sum_r \int d^3\mathbf{k} \left(\widetilde{\alpha}^{r\dagger}_{\mathbf{k}\sigma}(t)\widetilde{\alpha}^{r}_{\mathbf{k}\sigma}(t) - \widetilde{\beta}^{r\dagger}_{-\mathbf{k}\sigma}(t)\widetilde{\beta}^{r}_{-\mathbf{k}\sigma}(t)\right).$$

- The new flavor states

$$|\widetilde{\nu}^r_{\mathbf{k},\sigma}(t)\rangle = \widetilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}(t)|\widetilde{0}(t)\rangle_{e\mu}.$$

are locally eigenstates of a four momentum operator:

$$\begin{pmatrix} \widetilde{H}_\sigma(t) \\ \widetilde{\mathbf{P}}_\sigma(t) \end{pmatrix} |\widetilde{\nu}^r_{\mathbf{k},\sigma}(t)\rangle = \begin{pmatrix} \omega_{\mathbf{k},\sigma} \\ \mathbf{k} \end{pmatrix} |\widetilde{\nu}^r_{\mathbf{k},\sigma}(t)\rangle,$$

Poincaré structure

Define the Lorentz generators:

$$\widetilde{M}^{\lambda\rho}(t) = \int d^3\mathbf{x} \left(\widetilde{T}^{0\rho} x^\lambda - \widetilde{T}^{0\lambda} x^\rho\right) + \frac{1}{2}\int d^3\mathbf{x}\, \nu_f^\dagger \sigma^{\lambda\rho} \nu_f = \widetilde{M}_e^{\lambda\rho}(t) + \widetilde{M}_\mu^{\lambda\rho}(t),$$

where $\sigma^{\mu\nu} = -\frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

Algebra of *equal-time* commutators of the generators $(\sigma, \sigma' = e, \mu)$.

$$[\widetilde{P}_\sigma^\mu, \widetilde{P}_{\sigma'}^\nu] = 0 \quad ; \qquad [\widetilde{M}_\sigma^{\mu\nu}, \widetilde{P}_{\sigma'}^\lambda] = i\delta_{\sigma\sigma'} \left(\eta^{\mu\lambda} \widetilde{P}_\sigma^\nu - \eta^{\nu\lambda} \widetilde{P}_\sigma^\mu\right);$$

$$[\widetilde{M}_\sigma^{\mu\nu}, \widetilde{M}_{\sigma'}^{\lambda\rho}] = i\delta_{\sigma\sigma'} \left(\eta^{\mu\lambda} \widetilde{M}_\sigma^{\nu\rho} - \eta^{\nu\lambda} \widetilde{M}_\sigma^{\mu\rho} - \eta^{\mu\rho} \widetilde{M}_\sigma^{\nu\lambda} + \eta^{\nu\rho} \widetilde{M}_\sigma^{\mu\lambda}\right).$$

- The Poincaré structure is preserved in the interaction vertices.

Physical picture (optical analogy)

• Flavor neutrinos are (locally) on-shell particles, with masses:

$$m_e = m_1 \cos^2\theta + m_2 \sin^2\theta, \qquad m_\mu = m_2 \cos^2\theta + m_1 \sin^2\theta.$$

• Oscillations arise because of interaction with the external gauge field.

• Lorentz symmetry breaking is due to the external field.

• The vacuum acts as a sort of refractive medium ("*neutrino aether*") with respect to neutrinos.

• Optical analogy: flavor neutrinos as polarizations of the light, oscillations induced by birefringence*.

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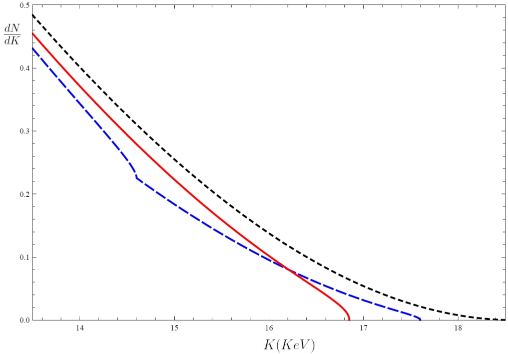
*C. Weinheimer, Prog. Part. Nucl. Phys., **64** (2010) 205.

Phenomenological consequences



The tail of the tritium β spectrum for:

- a massless neutrino (dotted line);

- fundamental flavor states (continuous line);

- superimposed prediction for 2 mass states (short-dashed line):

We used $m_e = 1.75$ KeV, $m_1 = 1$ KeV, $m_2 = 4$ KeV, $\theta = \pi/6$.

Thermodynamic analogy

Identify

$$F \equiv \widetilde{H}, \qquad T = g = \tan 2\theta$$

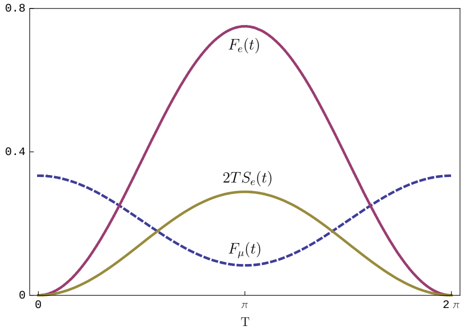
and write

$$H - F = TS,$$

$$S \;=\; \int d^3\mathbf{x}\, \bar{\nu}_f\, A_0 \nu_f \;=\; \frac{1}{2}\,\delta m \int d^3\mathbf{x}\, (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e).$$

• F is the energy that can be extracted from neutrinos through scattering experiments.

• Each of the two neutrinos can be considered as an open (dissipative) system.



Plot of expectation values on $|\nu_e(0)\rangle$ of $F_e(t)$, $F_\mu(t)$ and $2TS_e(t)$, as functions of dimensionless time $T = (\omega_2 - \omega_1)t$ and $\theta = \pi/6$. Scale on vertical axis is normalized to ω_μ.

Dynamical generation of flavor mixing

Dynamical generation of flavor mixing*

• The non trivial nature of flavor vacuum should result from the SSB process and the Higgs mechanism in the Standard Model;

• We consider dynamical symmetry breaking in a toy model with two flavors and quartic interaction term, as a generalization of Nambu and Jona-Lasinio model†;

• The approach of Umezawa, Takahashi and Kamefuchi for describing mass generation using inequivalent representations‡ is suitable for our purposes.

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*M.B., P.Jizba, G.Lambiase and N.Mavromatos, J. Phys. Conf. Ser. (2014); M.B., P.Jizba and L. Smaldone, work in progress;

†Y. Nambu and G. Jona-Lasinio, Phys. Rev. (1961);

‡H. Umezawa, Y. Takahashi and S. Kamefuchi, Ann. Phys. (1964)

Dynamical mass generation and inequivalent reps.

Consider a free Dirac field (at finite volume V):

$$\psi = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u_{\mathbf{k}} \, a^r_{\mathbf{k}} e^{-ik\cdot x} \; + \; v_{\mathbf{k}} \, b^{r\dagger}_{\mathbf{k}} e^{ik\cdot x} \right] , \qquad a^r_{\mathbf{k}}|0\rangle = b^r_{\mathbf{k}}|0\rangle = 0$$

The *same* field operator can be expanded as

$$\psi \; = \; \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k}}(\vartheta,\varphi) \alpha^r_{\mathbf{k}} \, e^{i\mathbf{k}\cdot\mathbf{x}} + v^r_{\mathbf{k}}(\vartheta,\varphi) \beta^{r\dagger}_{\mathbf{k}} \, e^{-i\mathbf{k}\cdot\mathbf{x}} \right],$$

with

$$\alpha^r_{\mathbf{k}} \;\; = \;\; \cos\vartheta^r_{\mathbf{k}} \, a^r_{\mathbf{k}} \; + \; \sin\vartheta^r_{\mathbf{k}} \, e^{i\varphi^r_k} \, b^{r\dagger}_{-\mathbf{k}}$$

$$\beta^r_{-\mathbf{k}} \;\; = \;\; \cos\vartheta^r_{\mathbf{k}} \, b^r_{-\mathbf{k}} \; - \; \sin\vartheta^r_{\mathbf{k}} \, e^{i\varphi^r_k} \, a^{r\dagger}_{\mathbf{k}}$$

and

$$u^r_{\mathbf{k}}(\vartheta,\phi) \; = \; u^r_{\mathbf{k}} \, \cos\vartheta_k \; + \; v^r_{-\mathbf{k}} \, e^{-i\varphi^r_k} \sin\vartheta_k \, ,$$

$$v^r_{\mathbf{k}}(\vartheta,\phi) \; = \; v^r_{\mathbf{k}} \, \cos\vartheta_k \; - \; u^r_{-\mathbf{k}} \, e^{i\varphi^r_k} \sin\vartheta_k \, .$$

The above is a Bogoliubov transformation, inducing inequivalent representations for different values of the parameters (ϑ, φ):

$$|0(\vartheta,\varphi)\rangle = \prod_{\mathbf{k},r} \Big[\cos\vartheta_k - e^{i\varphi_k^r} \sin\vartheta_k \; a_{\mathbf{k}}^{r\dagger} \, b_{-\mathbf{k}}^{r\dagger} \Big] |0\rangle$$

with $\alpha_{\mathbf{k}}^r \, |0(\vartheta,\varphi)\rangle \; = \; \beta_{\mathbf{k}}^r \, |0(\vartheta,\varphi)\rangle \; = \; 0 \, .$

V-limit for operator products

In the infinite volume limit, one has the following relations:

$$V\text{-lim}\left[\int d^3\boldsymbol{x}\,\bar\psi_\alpha(x)\psi_\beta(x)\right] \;=\; \int d^3\boldsymbol{x}\; :\bar\psi_\alpha(x)\psi_\beta(x): \;+\; \int d^3\boldsymbol{x}\, iS^-_{\alpha\beta}(\vartheta,\varphi)\,,$$

$$V\text{-lim}\left[\int d^3\boldsymbol{x}\,\bar\psi_\alpha(x)\psi_\beta(x)\bar\psi_\gamma(x)\psi_\delta(x)\right] \;=$$

$$= \; iS^-_{\alpha\beta}(\vartheta,\varphi)\int d^3\boldsymbol{x}\; :\bar\psi_\gamma(x)\psi_\delta(x): \;+ iS^+_{\gamma\delta}(\vartheta,\varphi)\int d^3\boldsymbol{x}\; :\bar\psi_\alpha(x)\psi_\beta(x):$$

$$+ \;\; iS^-_{\alpha\delta}(\vartheta,\varphi)\int d^3\boldsymbol{x}\; :\bar\psi_\gamma(x)\psi_\beta(x): \;+ iS^+_{\gamma\beta}(\vartheta,\varphi)\int d^3\boldsymbol{x}\; :\bar\psi_\alpha(x)\psi_\delta(x):$$

$$+\int d^3\boldsymbol{x}\sum_{\text{contractions}} S^+(\vartheta,\varphi)S^+(\vartheta,\varphi)\,.$$

$S^\pm_{\alpha\beta}(\theta,\varphi)$ are free two-point Wightman functions evaluated in $|0(\theta,\varphi)\rangle$:

$$iS^+_{\alpha\beta}(\vartheta,\varphi) \;=\; \langle 0(\vartheta,\varphi)|\bar\psi_\alpha(x)\psi_\beta(x)|0(\vartheta,\varphi)\rangle,$$

$$iS^-_{\alpha\beta}(\vartheta,\varphi) \;=\; \langle 0(\theta,\varphi)|\bar\psi_\alpha(x)\psi_\beta(x)|0(\vartheta,\varphi)\rangle$$

We consider the following hamiltonian:

$$H = H_0 + H_{\rm int} \,,$$

$$H_0 = \int d^3\mathbf{x}\, \bar\psi \,(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} \,+\, m)\,\psi \,,$$

$$H_{\rm int} = \lambda \int d^3\mathbf{x} \left[\left(\bar\psi\psi\right)^2 - \left(\bar\psi\gamma^5\psi\right)^2 \right] .$$

In the lowest order in the Yang-Feldman eq. the V-limit of H gives:

$$V\text{-lim}\,[H] = \bar H_0 + c - number.$$

with

$$\bar H_0 = H_0 + \delta H_0$$

$$\delta H_0 = \int d^3x \left(f\bar\psi\psi + ig\bar\psi\gamma_5\psi \right)$$

where f, g depend on the set of parameters (ϑ, φ):

$$f = \lambda\, C_s \,, \qquad g = \lambda\, C_p \,.$$

$$
\begin{aligned}
C_p &\equiv i \lim_{V\to\infty} \langle 0(\vartheta,\varphi)|\bar{\psi}(x)\gamma_5\psi(x)|0(\vartheta,\varphi)\rangle \\
&= \frac{2}{(2\pi)^3}\int d^3\mathbf{k}\, \sin 2\vartheta_k\;\; \sin\varphi_k
\end{aligned}
$$

$$
\begin{aligned}
C_s &\equiv \lim_{V\to\infty} \langle 0(\vartheta,\varphi)|\bar{\psi}(x)\psi(x)|0(\vartheta,\varphi)\rangle \\
&= -\frac{2}{(2\pi)^3}\int d^3\mathbf{k}\, \left[\frac{m}{\omega_k}\cos 2\vartheta_k \;-\; \frac{k}{\omega_k}\sin 2\vartheta_k \cos\varphi_k\right].
\end{aligned}
$$

We then require that $\bar{H}_0$ has the form of a free Hamiltonian:

$$\bar{H}_0 = \sum_r \int d^3\mathbf{k}\, E_\mathbf{k} \left(\alpha_\mathbf{k}^{r\dagger}\, \alpha_\mathbf{k}^r + \beta_\mathbf{k}^{r\dagger}\, \beta_\mathbf{k}^r \right) + W_0 .$$

with

$$E_\mathbf{k} = \sqrt{\mathbf{k}^2 + M^2} \quad ; \qquad W_0 = -2 \int d^3\mathbf{k}\, E_\mathbf{k} .$$

by fixing the Bogoliubov transformation parameters. One obtains:

$$\cos 2\vartheta_\mathbf{k}^r = \frac{1}{E_k} \left[\omega_k + f \frac{m}{\omega_k} \right]$$

$$\cos\varphi(\mathbf{k},\mathbf{r}) = -f \frac{k}{\omega_k} \frac{1}{\sqrt{g^2 + f^2 (k^2/\omega_k^2)}}$$

$$M^2 = (m+f)^2 + g^2 .$$

Two possibilities:

$$C_p = 0, \qquad M = m - \frac{2\lambda}{(2\pi)^3}\, M \int \frac{d^3\mathbf{k}}{E_k},$$

$$m = 0, \qquad 1 + \frac{2\lambda}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{E_k} = 0.$$

The second case is only allowed for $\lambda < 0$.

Dynamical generation of flavor mixing

– We consider the following hamiltonian:

$$H = H_0 + H_{int}$$

$$H_0 = \int d^3x \bar{\Psi} \left(-i\gamma^i \partial_i + M_0 \right) \Psi$$

with $\Psi^T = (\psi_{\rm I}, \psi_{\rm II})$ and $M_0 = diag(m_{\rm I}, m_{\rm II})$.

– The interaction Hamiltonian $H_{\rm int}$ can be assumed in the generic form

$$\mathcal{H}_{\rm int} = \left(\bar{\boldsymbol{\psi}} \, \Gamma \, \boldsymbol{\psi} \right) \left(\bar{\boldsymbol{\psi}} \, \Gamma' \, \boldsymbol{\psi} \right),$$

where Γ and Γ' are some doublet spinor matrices.

– The V-limit renormalization term $\delta\mathcal{H}_0$ has the following structure

$$\begin{aligned} \delta\mathcal{H}_0 &= \delta\mathcal{H}_0^{\rm I} + \delta\mathcal{H}_0^{\rm II} + \delta\mathcal{H}_{\rm mix} \\ &= f_{\rm I} \, \bar{\psi}_{\rm I} \psi_{\rm I} + f_{\rm II} \, \bar{\psi}_{\rm II} \psi_{\rm II} + h \left(\bar{\psi}_{\rm I} \psi_{\rm II} + \bar{\psi}_{\rm II} \psi_{\rm I} \right) . \end{aligned}$$

Generalized Bogoliubov transformation

– We consider the 4×4 canonical transformation

$$\begin{pmatrix} \alpha_A \\ \alpha_B \\ \beta_A^\dagger \\ \beta_B^\dagger \end{pmatrix} = \begin{pmatrix} c_\theta\,\rho_{A\,\mathrm{I}} & s_\theta\,\rho_{A\,\mathrm{II}} & c_\theta\,\lambda_{A\,\mathrm{I}} & s_\theta\,\lambda_{A\,\mathrm{II}} \\ -s_\theta\,\rho_{B\,\mathrm{I}} & c_\theta\,\rho_{B\,\mathrm{II}} & -s_\theta\,\lambda_{B\,\mathrm{I}} & c_\theta\,\lambda_{B\,\mathrm{II}} \\ c_\theta\,\lambda_{A\,\mathrm{I}} & s_\theta\,\lambda_{A\,\mathrm{II}} & c_\theta\,\rho_{A\,\mathrm{I}} & s_\theta\,\rho_{A\,\mathrm{II}} \\ -s_\theta\,\lambda_{B\,\mathrm{I}} & c_\theta\,\lambda_{B\,\mathrm{II}} & -s_\theta\,\rho_{B\,\mathrm{I}} & c_\theta\,\rho_{B\,\mathrm{I}} \end{pmatrix} \begin{pmatrix} a_{\mathrm{I}} \\ a_{\mathrm{II}} \\ b_{\mathrm{I}}^\dagger \\ b_{\mathrm{II}}^\dagger \end{pmatrix}$$

where $c_\theta \equiv \cos\theta$, $s_\theta \equiv \sin\theta$ and

$$\rho_{ab} \equiv \cos\frac{\chi_a - \chi_b}{2}, \quad \lambda_{ab} \equiv \sin\frac{\chi_a - \chi_b}{2}, \quad \chi_a \equiv \cot^{-1}\left[\frac{k}{m_a}\right],\ a, b = I, II, A, B.$$

Thus we have three parameters (θ, m_A, m_B) to fix in terms of $(f_{\mathrm{I}}, f_{\mathrm{II}}, h)$ in order to diagonalize the Hamiltonian.

A possible representation is obtained by a partial diagonalization of $\bar{H}_0$, leaving untouched $\delta\mathcal{H}_{mix}$:

$$\bar{\mathcal{H}}_0 \;=\; \sum_{\sigma=e,\mu} \bar{\psi}_\sigma \left(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} \;+\; m_\sigma\right)\psi_\sigma \;+\; h\left(\bar{\psi}_e\psi_\mu \;+\; \bar{\psi}_\mu\psi_e\right).$$

Such a representation is obtained by setting

$$\begin{aligned} \theta &\;\to\; 0\,, \\ m_A &\;\to\; m_e \;\equiv\; m_{\rm I} + f_{\rm I}\,, \\ m_B &\;\to\; m_\mu \;\equiv\; m_{\rm II} + f_{\rm II}\,. \end{aligned}$$

The vacuum is denoted as

$$|0(\theta = 0, m_e, m_\mu)\rangle \;\equiv\; |0\rangle_{e\mu}\,,$$

In this representation we have

$${}_{e,\mu}\langle 0|\bar{H}_0|0\rangle_{e,\mu} \;=\; -2\int d^3\mathbf{k}\;\left(\sqrt{k^2 \;+\; m_e^2} \;+\; \sqrt{k^2 \;+\; m_\mu^2}\right),$$

since ${}_{e,\mu}\langle 0|\delta\mathcal{H}_{\rm mix}|0\rangle_{e,\mu} \;=\; 0.$

Complete diagonalization

Another possibility is to require that $\bar{\mathcal{H}}_0$ becomes fully diagonal in two fermion fields, ψ_1 and ψ_2, with masses m_1 and m_2:

$$\bar{\mathcal{H}}_0 \;=\; \sum_{j=1,2} \bar{\psi}_j \left(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} \;+\; m_j\right)\psi_j \,.$$

The condition for the complete diagonalization is found to be:

$$\theta \quad\rightarrow\quad \bar{\theta} \;\equiv\; \frac{1}{2}\,\tan^{-1}\left[\frac{2h}{m_\mu - m_e}\right],$$

$$m_{_{A,B}} \quad\rightarrow\quad m_{1,2} \;\equiv\; \frac{1}{2}\left(m_e + m_\mu \mp \sqrt{(m_\mu - m_e)^2 + 4h^2}\right),$$

where we introduced the notation $m_e = m_{\rm I} + f_{\rm I}$, $m_\mu = m_{\rm II} + f_{\rm II}$.

We set

$$|0(\bar{\theta}, m_1, m_2)\rangle \;\equiv\; |0\rangle_{1,2}\,,$$

The vev of the Hamiltonian in this representation has the form:

$${}_{1,2}\langle 0|\bar{H}_0|0\rangle_{1,2} \;=\; -2\int d^3\mathbf{k}\;\left(\sqrt{k^2 \;+\; m_1^2} \;+\; \sqrt{k^2 \;+\; m_2^2}\right).$$

Consider the Lagrangian

$$\mathcal{L} \;=\; i\bar{\boldsymbol{\psi}}\gamma^{\mu}\partial_{\mu}\boldsymbol{\psi} \;+\; U(\bar{\boldsymbol{\psi}},\boldsymbol{\psi})$$

where

$$\boldsymbol{\psi} \;=\; \begin{bmatrix}\psi_{\text{I}}\\ \psi_{\text{II}}\end{bmatrix}\,.$$

$U(\bar{\boldsymbol{\psi}},\boldsymbol{\psi})$ is assumed to be invariant under chiral transformations $U(2)_V \times U(2)_A$:

$$\mathbf{g} \;=\; (g, g_5)\;,$$

$$g \;=\; e^{i\omega_\alpha \frac{\sigma^\alpha}{2}}\,, \qquad\qquad g_5 \;=\; e^{i\omega_\alpha \frac{\sigma^\alpha\gamma^5}{2}}\,, \quad \alpha = 0,1,2,3$$

so the entire Lagrangian is invariant as well.

Chiral Symmetry Breaking

The vector and axial Noether charges are:

$$J_\mu^\alpha = \overline{\psi}\gamma_\mu \frac{\sigma^\alpha}{2}\psi \,,$$

$$J_{5\mu}^\alpha = \overline{\psi}\gamma_\mu\gamma_5 \frac{\sigma^\alpha}{2}\psi \,.$$

If we add a diagonal mass term

$$\mathcal{L}_M = -m\overline{\psi}\psi$$

the conservation law of axial currents is explicitly broken:

$$\partial^\mu J_\mu^\alpha = 0 \,,$$

$$\partial^\mu J_{5\mu}^\alpha = i\overline{\psi}\,\gamma_5 m\,\psi \,,$$

Isospin Symmetry Breaking

Adding a mass-shift term

$$\mathcal{L}_{\Delta m} \;=\; -\overline{\boldsymbol{\psi}} \begin{bmatrix} -\Delta m & 0 \\ 0 & \Delta m \end{bmatrix} \boldsymbol{\psi}$$

the isospin symmetry is broken to $U(1)_V^0 \times U(1)_V^3$ (the subscript index indicates the generator)

$$\begin{aligned} \partial^\mu J_\mu^0 &= \partial^\mu J_\mu^3 \;=\; 0\,, \\ \partial^\mu J_\mu^1 &= \frac{\Delta m}{2}\,\overline{\boldsymbol{\psi}}\,\sigma_2\,\boldsymbol{\psi}\,, \\ \partial^\mu J_\mu^2 &= -\frac{\Delta m}{2}\,\overline{\boldsymbol{\psi}}\,\sigma_1\,\boldsymbol{\psi}\,. \end{aligned}$$

Family lepton number nonconservation

Finally we add to the Lagrangian, an off-diagonal term

$$\mathcal{L}_h \;=\; -\overline{\boldsymbol{\psi}} \begin{bmatrix} 0 & h \\ h & 0 \end{bmatrix} \boldsymbol{\psi} \,.$$

The current evolution are now

$$\begin{aligned} \partial^\mu J^0_\mu &= 0 \,, \\ \partial^\mu J^1_\mu &= \frac{\Delta m}{2} \overline{\boldsymbol{\psi}} \, \sigma_2 \, \boldsymbol{\psi} \,, \\ \partial^\mu J^2_\mu &= -\frac{1}{2} \overline{\boldsymbol{\psi}} \left[h\sigma_3 + \Delta m \sigma_1 \right] \boldsymbol{\psi} \,, \\ \partial^\mu J^3_\mu &= \frac{h}{2} \overline{\boldsymbol{\psi}} \, \sigma_2 \, \boldsymbol{\psi} \,, \end{aligned}$$

Conservation of the total flavor charge Q^0.

Order parameters

Dynamical generation of mixing occurs if*

$$U(2)_V \times U(2)_A \longrightarrow U(1)_V^0\,,$$

at the ground state level. SSB is characterized by the existence of some (quasi)-local operators Φ_i so that

$$\langle\Omega|\left[Q^\alpha(0),\Phi_i(0)\right]|\Omega\rangle \;=\; \langle\Omega|\varphi_i^\alpha|\Omega\rangle \;\neq\; 0\,,$$

on some *dressed* vacuum. φ_i^α are called *order parameters*. We look at order parameters of the form $\overline{\psi}_i\psi_j \pm \overline{\psi}_k\psi_l$ with $i,j,k,l = \mathrm{I},\mathrm{II}$.

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*M. Blasone, P. Jizba, L. S., in preparation (2017).

Patterns of SSB

Symmetry Group	Mass Term	Order Parameter	Broken Charges
$U(2)_V \times U(2)_A$	$m = 0$		
$U(2)_V$	$m \neq 0; \quad \Delta m = 0$	$\langle \overline{\psi}_{\rm I}\psi_{\rm I} + \overline{\psi}_{\rm II}\psi_{\rm II} \rangle \neq 0$	Q_5^α
$U(1)_V^0 \times U(1)_V^3$	$m \neq 0; \quad \Delta m \neq 0$	$\langle \overline{\psi}_{\rm I}\psi_{\rm I} \pm \overline{\psi}_{\rm II}\psi_{\rm II} \rangle \neq 0$	$Q_5^\alpha;\ Q^1;\ Q^2$
$U(1)_V^0$	$m \neq 0; \quad \Delta m \neq 0$ $h \neq 0$	$\langle \bar{\psi}_{\rm I}\psi_{\rm I} \pm \bar{\psi}_{\rm II}\psi_{\rm II} \rangle \neq 0$ $\langle \overline{\psi}_{\rm I}\psi_{\rm II} + \overline{\psi}_{\rm II}\psi_{\rm I} \rangle \neq 0$	$Q_5^\alpha;\ Q^1;\ Q^2;\ Q^3$

Field mixing in Rindler spacetime

Boson mixing in QFT (Minkowski spacetime)

Boson mixing transformations:†

$$\phi_A(x) = \phi_1(x)\,\cos\theta + \phi_2(x)\,\sin\theta = G_\theta^{-1}(t)\,\phi_1(x)\,G_\theta(t)$$

$$\phi_B(x) = -\phi_1(x)\,\sin\theta + \phi_2(x)\,\cos\theta = G_\theta^{-1}(t)\,\phi_2(x)\,G_\theta(t)$$

where

$$G_\theta(t) = \exp\left[-i\theta\int d^3\mathbf{x}\left(\pi_1\phi_2 - \phi_1^\dagger\pi_2^\dagger - \pi_2\phi_1 + \phi_2^\dagger\pi_1^\dagger\right)\right].$$

Mapping between the Fock spaces $\mathcal{H}_{1,2}$ and $\mathcal{H}_{A,B}$ (at volume V)

$$|0(\theta,t)\rangle_{A,B} = G_\theta^{-1}(t)\;|0\rangle_{1,2}\;.$$

Unitary inequivalence

$$\lim_{V\to\infty}\;{}_{1,2}\langle 0|0(\theta,t)\rangle_{A,B} = 0, \quad \forall t.$$

† M.B., A.Capolupo, O.Romei and G.Vitiello, Phys. Rev. **D** (2001)

Field expansions in plane-wave basis $(i = 1, 2)$

- Fields with definite mass

$$\phi_i = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2\omega_{k,i}}} \left(a_{\mathbf{k},i} e^{-i\omega_{k,i} t} + \bar{a}^\dagger_{-\mathbf{k},i} e^{i\omega_{k,i} t} \right),$$

with

$$a_{\mathbf{k},i}|0\rangle_{1,2} = 0, \qquad \left[a_{\mathbf{k},i}, a^\dagger_{\mathbf{k}',j} \right] = \delta^3(\mathbf{k} - \mathbf{k}')\delta_{ij}.$$

- Fields with definite flavor $\big((\chi, j) = (A, 1), (B, 2)\big)$

$$\phi_\chi = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{2\omega_{k,j}}} \left(a_{\mathbf{k},\chi} e^{-i\omega_{k,j} t} + \bar{a}^\dagger_{-\mathbf{k},\chi} e^{i\omega_{k,j} t} \right),$$

with

$$a_{\mathbf{k},\chi}(t)|0(t)\rangle_{A,B} = 0, \qquad \left[a_{\mathbf{k},\chi}(t), a^\dagger_{\mathbf{k}',\sigma}(t) \right] = \delta^3(\mathbf{k} - \mathbf{k}')\delta_{\chi\sigma} \quad \forall t$$

- Annihilator for the flavor vacuum $|0(\theta,t)\rangle_{A,B}$

$$a_{\mathbf{k},A}(t) = \cos\theta\, a_{\mathbf{k},1} \;+\; \sin\theta\, \left(\rho_{12}^{\mathbf{k}\,*}(t)\, a_{\mathbf{k},2} \;+\; \lambda_{12}^{\mathbf{k}}(t)\, \bar{a}^{\dagger}_{-\mathbf{k},2}\right)$$

- mixing Bogoliubov coefficients:

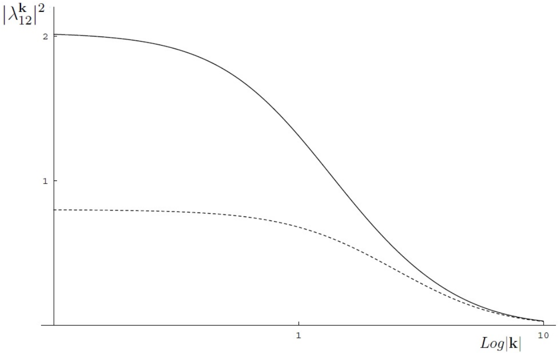
$$\rho_{12}^{\mathbf{k}}(t) = |\rho_{12}^{\mathbf{k}}|\; e^{i(\omega_{k,2}-\omega_{k,1})t}, \qquad \lambda_{12}^{\mathbf{k}}(t) = |\lambda_{12}^{\mathbf{k}}|\; e^{i(\omega_{k,2}+\omega_{k,1})t}$$

$$|\lambda_{12}^{\mathbf{k}}| = \frac{1}{2}\left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} - \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}}\right)$$

with $|\rho_{12}^{\mathbf{k}}|^2 - |\lambda_{12}^{\mathbf{k}}|^2 = 1$.

- Condensation density for mixed bosons

$${}_{A,B}\langle 0(t)|a^{\dagger}_{\mathbf{k},i}\, a_{\mathbf{k},i}|0(t)\rangle_{A,B} = \sin^2\theta\; |\lambda_{12}^{\mathbf{k}}|^2$$



Condensation density as a function of $Log|\mathbf{k}|$ for sample values of m_1 and m_2. Solid line: $m_1 = 10, m_2 = 100$. Dashed line: $m_1 = 1, m_2 = 100$.

Uniformly accelerated observer: Rindler metric

- Rindler coordinates

$$x^0 = \xi \sinh \eta, \quad x^1 = \xi \cosh \eta$$

- Rindler ($\mathscr{R}$) vs Minkowski ($\mathscr{M}$)

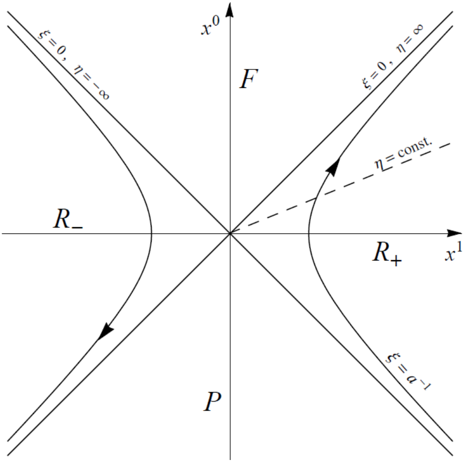
$$ds^2 = \left(dx^0\right)^2 - (dx^1)^2 - (d\vec{x})^2 \rightarrow$$

$$\rightarrow ds^2 = \xi^2 d\eta^2 - d\xi^2 - (d\vec{x})^2$$

- Worldline of a Rindler observer

$$\eta = a\tau, \quad \xi = \text{const} \equiv a^{-1}, \quad \vec{x} = \text{const}$$

with proper acceleration a.



Free boson field in hyperbolic basis

- Free field quantization in $\mathscr{M}$ in hyperbolic basis

$$\phi = \int d^3\kappa \sum_{\sigma} \left\{ d_{\kappa}^{(\sigma)} \; \widetilde{U}_{\kappa}^{(\sigma)} + \bar{d}_{\kappa}^{(\sigma)\,\dagger} \; \widetilde{U}_{\kappa}^{(\sigma)*} \right\}, \quad \kappa = (\Omega, \vec{k})$$

where

$$\widetilde{U}_{\kappa}^{(\sigma)} = \frac{2^{1/2}\, e^{\sigma\pi\Omega/2}}{(2\pi)^{n/2}} \, K_{i\sigma\Omega}(\mu_k \xi)\, e^{i\left(\vec{k}\cdot\vec{x} - \sigma\Omega\eta\right)}$$

and

$$d_{\kappa}^{(\sigma)} = \int_{-\infty}^{+\infty} \frac{dk_1}{(2\pi\omega_k)^{1/2}} \left(\frac{\omega_k + k_1}{\omega_k - k_1} \right)^{i\sigma\Omega/2} a_{k_1, \vec{k}}, \qquad \left[d_{\kappa}^{(\sigma)} \,,\, d_{\kappa'}^{(\sigma')\,\dagger} \right] = \delta_{\sigma\sigma'} \, \delta^3 \,(\kappa - \kappa').$$

- $d_{\kappa}^{(\sigma)}$ and $a_{\mathbf{k}}$ annihilate the same Minkowski vacuum $|0_M\rangle$

$$d_{\kappa}^{(\sigma)} |0_M\rangle = a_{\mathbf{k}} |0_M\rangle = 0.$$

Free field quantization in Rindler spacetime

–Free field quantization in $\mathscr{R}$ in Rindler modes*

$$\phi = \int d^3\kappa \sum_\sigma \left\{ b_\kappa^{(\sigma)}\, u_\kappa^{(\sigma)} + \bar{b}_\kappa^{(\sigma)\,\dagger}\, u_\kappa^{(\sigma)*} \right\}$$

where

$$u_\kappa^{(\sigma)} = \frac{\theta(\sigma\xi)}{2\Omega\sqrt{2\pi}}\, h_\kappa^{(\sigma)}(\xi)\, e^{i\left(\vec{k}\cdot\vec{x}-\sigma\Omega\eta\right)}$$

and

$$\left[b_\kappa^{(\sigma)}\,,\, b_{\kappa'}^{(\sigma')\,\dagger}\right] = \delta_{\sigma\sigma'}\,\delta^3(\kappa-\kappa'), \qquad b_\kappa^{(\sigma)}|0_R\rangle = 0.$$

$|0_R\rangle$ is the Rindler vacuum.

*S. Takagi, Prog. Theor. Phys. Suppl.(1986)

Minkowski quantization:

$$\phi = \int d^3\kappa \sum_\sigma \left\{ d_\kappa^{(\sigma)} \, \widetilde{U}_\kappa^{(\sigma)} + \bar{d}_\kappa^{(\sigma)\,\dagger} \, \widetilde{U}_\kappa^{(\sigma)*} \right\}$$

Rindler quantization:

$$\phi = \int d^3\kappa \sum_\sigma \left\{ b_\kappa^{(\sigma)} \, u_\kappa^{(\sigma)} + \bar{b}_\kappa^{(\sigma)\,\dagger} \, u_\kappa^{(\sigma)*} \right\}$$

↘ ↙

Thermal Bogoliubov transformation

$$b_\kappa^{(\sigma)} = \sqrt{1 + N(\Omega)}\, d_\kappa^{(\sigma)} + \sqrt{N(\Omega)}\, \bar{d}_{\tilde{\kappa}}^{(-\sigma)\,\dagger}$$

where $N(\Omega) = \left(e^{2\pi\Omega} - 1\right)^{-1}$ and $\tilde{\kappa} = (\Omega, -\vec{k})$.

- Rindler vacuum $|0_R\rangle$ differs from Minkowski vacuum $|0_M\rangle$

$$\langle 0_M | b_\kappa^{(\sigma)\,\dagger} \, b_{\kappa'}^{(\sigma')} | 0_M \rangle = N(\Omega)\, \delta_{\sigma\sigma'}\, \delta^3\left(\kappa - \kappa'\right).$$

- For a Rindler observer, $|0_M\rangle$ is seen to be equivalent to a thermal bath with temperature $T = \frac{a}{2\pi}$ †.

† W. G. Unruh, Phys. Rev. D (1976)

Mixing and thermal Bogoliubov transformations

Two Bogoliubov transformations

$$\phi_1, \phi_2 \xrightarrow{\text{mixing Bogol. }(\theta)} \phi_A, \phi_B \;\Rightarrow\; \text{condensate in } |0_{A,B}\rangle,$$

$$\phi_{\mathscr{R}} \xrightarrow{\text{thermal Bogol.}\,(a)} \phi_{\mathscr{M}} \;\Rightarrow\; \text{condensate in } |0_M\rangle.$$

• How do these two transformations combine when field mixing in accelerated frames is studied?*

*M. Blasone, G. Lambiase and G. Luciano, Phys. Rev D (2017).

Field mixing in Minkowski spacetime: hyperbolic basis

$$\phi_A(x) = \phi_1(x)\,\cos\theta + \phi_2(x)\,\sin\theta,$$

$$\phi_B(x) = -\phi_1(x)\,\sin\theta + \phi_2(x)\,\cos\theta$$

- Hyperbolic expansions of definite mass fields

$$\phi_i = \int d^3\kappa \sum_\sigma \left\{ d^{(\sigma)}_{\kappa,i}\,\widetilde{U}^{(\sigma)}_{\kappa,i} + \bar{d}^{(\sigma)\,\dagger}_{\kappa,i}\,\widetilde{U}^{(\sigma)*}_{\kappa,i} \right\}, \quad i = (1,2)$$

- Hyperbolic expansions of definite flavor fields

$$\phi_\chi = \int d^3\kappa \sum_\sigma \left\{ d^{(\sigma)}_{\kappa,\chi}\,\widetilde{U}^{(\sigma)}_{\kappa,j} + \bar{d}^{(\sigma)\,\dagger}_{\kappa,\chi}\,\widetilde{U}^{(\sigma)*}_{\kappa,j} \right\}, \quad (\chi, j) = (A,1), (B,2).$$

Flavor annihilator in hyperbolic basis

$$d^{(\sigma)}_{\kappa,A} = \cos\theta\, d^{(\sigma)}_{\kappa,1} + \sin\theta \int_0^{+\infty} d\Omega' \sum_{\sigma'} \left(d^{(\sigma')}_{(\Omega',\vec{k}),2}\, \mathcal{A}^{(\sigma,\sigma')\,*}_{(\Omega,\Omega'),\,\vec{k}} + \bar{d}^{(\sigma')\dagger}_{(\Omega',-\vec{k}),2}\, \mathcal{B}^{(\sigma,\sigma')}_{(\Omega,\Omega'),\,\vec{k}} \right),$$

where

$$\mathcal{A}^{(\sigma,\sigma')\,*}_{(\Omega,\Omega'),\,\vec{k}} = \int_{-\infty}^{+\infty} \frac{dk_1}{4\pi} \left[\left(\frac{1}{\omega_{\mathbf{k},1}} + \frac{1}{\omega_{\mathbf{k},2}} \right) \left(\frac{\omega_{\mathbf{k},1} + k_1}{\omega_{\mathbf{k},1} - k_1} \right)^{i\frac{\sigma\Omega}{2}} \left(\frac{\omega_{\mathbf{k},2} + k_1}{\omega_{\mathbf{k},2} - k_1} \right)^{-i\frac{\sigma'\Omega'}{2}} e^{i(\omega_{\mathbf{k},1} - \omega_{\mathbf{k},2})t} \right]$$

$$\mathcal{B}^{(\sigma,\sigma')}_{(\Omega,\Omega'),\,\vec{k}} = \int_{-\infty}^{+\infty} \frac{dk_1}{4\pi} \left[\left(\frac{1}{\omega_{\mathbf{k},2}} - \frac{1}{\omega_{\mathbf{k},1}} \right) \left(\frac{\omega_{\mathbf{k},1} + k_1}{\omega_{\mathbf{k},1} - k_1} \right)^{i\frac{\sigma\Omega}{2}} \left(\frac{\omega_{\mathbf{k},2} + k_1}{\omega_{\mathbf{k},2} - k_1} \right)^{-i\frac{\sigma'\Omega'}{2}} e^{i(\omega_{\mathbf{k},1} + \omega_{\mathbf{k},2})t} \right]$$

Non-trivial resolution of these integrals...

Field mixing in Rindler spacetime

- Rindler expansions of definite flavor fields in $\mathscr{R}$

$$\phi_\chi = \int d^3\kappa \sum_\sigma \left\{ b^{(\sigma)}_{\kappa,\chi}\, u^{(\sigma)}_{\kappa,j} + \bar{b}^{(\sigma)\,\dagger}_{\kappa,\chi}\, u^{(\sigma)*}_{\kappa,j} \right\}, \quad (\chi, j) = (A,1), (B,2)$$

- Mixing + thermal Bogoliubov transformation

$$b^{(\sigma)}_{\kappa,A} = \sqrt{(1+N(\Omega))}\, d^{(\sigma)}_{\kappa,A} + \sqrt{N(\Omega)}\, \bar{d}^{(-\sigma)\dagger}_{\tilde{\kappa},A}\,.$$

Condensation density of Rindler mixed particles in $|0_M\rangle$

$$\mathcal{N}_R(\theta)\Big|_0 = N_R(\Omega)\,\delta^3(\kappa-\kappa') + \sin^2\theta\Big[F(\Omega,\Omega')\,N_{\mathcal{BB}} + G(\Omega,\Omega')\,N_{\mathcal{AB}}\Big]\,\delta^2(\vec{k}-\vec{k}'),$$

with

$$F(\Omega,\Omega') \equiv \sqrt{N_R(\Omega)\,N_R(\Omega')} + \sqrt{(1+N_R(\Omega))\,(1+N_R(\Omega'))}\,,$$

$$G(\Omega,\Omega') \equiv \sqrt{1+N_R(\Omega)}\,\sqrt{N_R(\Omega')} + \sqrt{N_R(\Omega)}\,\sqrt{1+N_R(\Omega')}\,,$$

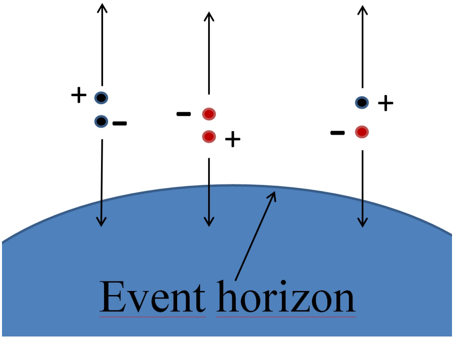
– Combination of the Unruh condensation density $N(\Omega)$ and mixing condensate $N_\theta(\Omega)$ ⇒ Geometry of spacetime as possible source of field mixing?

- Modification of the Bose-Einstein distribution.

Physical picture of the modification

No mixing: $|0_M\rangle$ is populated by Rindler particle/antiparticle pairs of the same type $\Rightarrow$ Particle radiation is generated by the corresponding one-type antiparticles falling into the event horizon $\Rightarrow$ *Thermal distribution.*

Mixing: Vacuum is a condensate of particle/antiparticle pairs of different types. $\Rightarrow$ Particle radiation is generated by both types of antiparticles falling into the event horizon $\Rightarrow$ Increase of the entropy and *modification* of the B-E distribution.



Modification of the B-E distribution due to the mixing. Different colours correspond to different types of particle.

Conclusions and Perspectives

Conclusions and Perspectives

- Mixing transformations are not trivial in Q.F.T. (not just a rotation!) $\Rightarrow$ inequivalent representations.

- The vacuum for mixed fields has the structure of a $SU(N)$ generalized coherent state (condensate of particle-antiparticle pairs).

- Condensate structure of the flavor vacuum $\Rightarrow$ dynamical origin of mixing.

- Lorentz invariance violation (?)

Conclusions and Perspectives

- Application of functional methods has suggested the study of appearance of inequivalent representations in path integrals[†].

- Neutrino oscillations in curved backgrounds;

- Entanglement in neutrino states: neutrino oscillations as a resource for quantum information;

- Geometric phases;

[†] M. B., P. Jizba, L. Smaldone, Ann. Phys. (2017).

Three-flavor fermion mixing[‡]

Mixing relations:

$$\Psi_f(x) \; = \; \mathbf{M}\, \Psi_m(x)$$

where $\Psi_f^T = (\nu_e, \nu_\mu, \nu_\tau)$, $\Psi_m^T = (\nu_1, \nu_2, \nu_3)$ and

$$\mathbf{M} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

with $c_{ij} = \cos\theta_{ij}, s_{ij} = \sin\theta_{ij}$

[‡] M.B., A.Capolupo and G.Vitiello, Phys. Rev. **D** (2002)

We have:

$$\nu_\sigma^\alpha(x) = G_\theta^{-1}(t)\,\nu_i^\alpha(x)\,G_\theta(t),$$

where $(\sigma, i) = (e,1), (\mu,2), (\tau,3)$, and

$$G_\theta(t) \quad = \quad G_{2_3}(t)G_{1_3}(t)G_{12}(t)$$

$$G_{12}(t) \quad = \quad exp\left[\theta_{12}\int d^3\mathbf{x}(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x))\right],$$

$$G_{13}(t) \quad = \quad exp\left[\theta_{13}\int d^3\mathbf{x}(\nu_1^\dagger(x)\nu_3(x)e^{-i\delta} - \nu_3^\dagger(x)\nu_1(x)e^{i\delta})\right],$$

$$G_{23}(t) \quad = \quad exp\left[\theta_{23}\int d^3\mathbf{x}(\nu_2^\dagger(x)\nu_3(x) - \nu_3^\dagger(x)\nu_2(x))\right],$$

Flavor vacuum:

$$|0\rangle_f \;=\; G_\theta^{-1}(t)\;|0\rangle_m$$

Flavor annihilation operators:

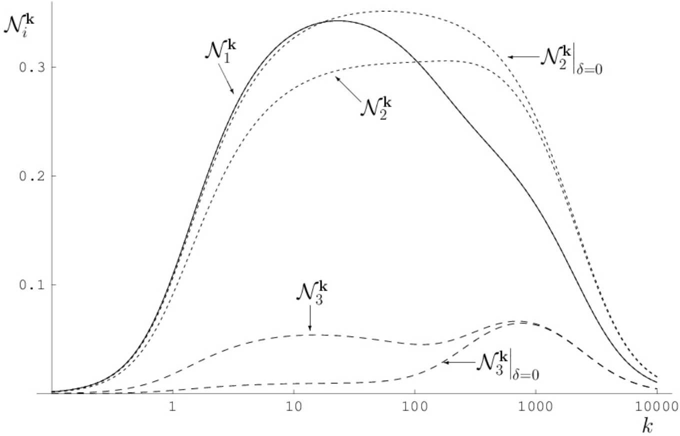
$$\alpha^r_{\mathbf{k},e} = c_{12}c_{13}\ \alpha^r_{\mathbf{k},1} + s_{12}c_{13}\left(U^{\mathbf{k}*}_{12}\ \alpha^r_{\mathbf{k},2} + \epsilon^r V^{\mathbf{k}}_{12}\ \beta^{r\dagger}_{-\mathbf{k},2}\right) + e^{-i\delta}\ s_{13}\left(U^{\mathbf{k}*}_{13}\ \alpha^r_{\mathbf{k},3} + \epsilon^r V^{\mathbf{k}}_{13}\ \beta^{r\dagger}_{-\mathbf{k},3}\right),$$

$$\alpha^r_{\mathbf{k},\mu} = \left(c_{12}c_{23} - e^{i\delta}\ s_{12}s_{23}s_{13}\right)\ \alpha^r_{\mathbf{k},2} - \left(s_{12}c_{23} + e^{i\delta}\ c_{12}s_{23}s_{13}\right)\left(U^{\mathbf{k}}_{12}\ \alpha^r_{\mathbf{k},1} - \epsilon^r V^{\mathbf{k}}_{12}\ \beta^{r\dagger}_{-\mathbf{k},1}\right)$$
$$+ s_{23}c_{13}\left(U^{\mathbf{k}*}_{23}\ \alpha^r_{\mathbf{k},3} + \epsilon^r V^{\mathbf{k}}_{23}\ \beta^{r\dagger}_{-\mathbf{k},3}\right),$$

$$\alpha^r_{\mathbf{k},\tau} = c_{23}c_{13}\ \alpha^r_{\mathbf{k},3} - \left(c_{12}s_{23} + e^{i\delta}\ s_{12}c_{23}s_{13}\right)\left(U^{\mathbf{k}}_{23}\ \alpha^r_{\mathbf{k},2} - \epsilon^r V^{\mathbf{k}}_{23}\ \beta^{r\dagger}_{-\mathbf{k},2}\right)$$
$$+ \left(s_{12}s_{23} - e^{i\delta}\ c_{12}c_{23}s_{13}\right)\left(U^{\mathbf{k}}_{13}\ \alpha^r_{\mathbf{k},1} - \epsilon^r V^{\mathbf{k}}_{13}\ \beta^{r\dagger}_{-\mathbf{k},1}\right)$$

and similar ones for antiparticles ($\delta \to -\delta$).

Condensation densities



Condensation densities $\mathcal{N}_i^{\mathbf{k}}$ for sample values of masses and mixings

Parameterizations of mixing matrix

$$\nu_\sigma^\alpha(x) = G_\theta^{-1}(t)\, \nu_i^\alpha(x)\, G_\theta(t),$$

Define the more general generators:

$$G_{12} \equiv \exp\left[\theta_{12} \int d^3x \left(\nu_1^\dagger \nu_2 e^{-i\delta_2} - \nu_2^\dagger \nu_1 e^{i\delta_2}\right)\right]$$

$$G_{1_3} \equiv \exp\left[\theta_{1_3} \int d^3x \left(\nu_1^\dagger \nu_{_3} e^{-i\delta_5} - \nu_{_3}^\dagger \nu_1 e^{i\delta_5}\right)\right]$$

$$G_{2_3} \equiv \exp\left[\theta_{2_3} \int d^3x \left(\nu_2^\dagger \nu_{_3} e^{-i\delta_7} - \nu_{_3}^\dagger \nu_2 e^{i\delta_7}\right)\right]$$

There are six different matrices obtained by permutations of the above generators.

We can obtain all possible parameterizations of the matrix by setting to zero two of the phases and permuting rows/columns.

Currents and charges for 3-flavor fermion mixing

Lagrangian for three free Dirac fields with different masses

$$\mathcal{L}(x) \,=\, \bar{\Psi}_m(x)\,(i\,\not{\partial} - M_d)\,\Psi_m(x)$$

where $\Psi_m^T = (\nu_1, \nu_2, \nu_3)$ and $M_d = diag(m_1, m_2, m_3)$.

The $SU(3)$ transformations:

$$\Psi'_m(x) \,=\, e^{i\alpha_j \lambda_j/2}\,\Psi_m(x) \quad ; \qquad j = 1,..,8$$

with α_j real constants, and λ_j the Gell-Mann matrices, give the currents:

$$J^\mu_{m,j}(x) \,=\, \frac{1}{2}\bar{\Psi}_m(x)\,\gamma^\mu\,\lambda_j\,\Psi_m(x)$$

The combinations:

$$Q_1 \equiv \frac{1}{3}Q + Q_{m,3} + \frac{1}{\sqrt{3}}Q_{m,8},$$

$$Q_2 \equiv \frac{1}{3}Q - Q_{m,3} + \frac{1}{\sqrt{3}}Q_{m,8}$$

$$Q_3 \equiv \frac{1}{3}Q - \frac{2}{\sqrt{3}}Q_{m,8}$$

$$Q_i = \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i}\beta^r_{-\mathbf{k},i} \right), \; i = 1, 2, 3.$$

are the Noether charges for the fields ν_i with $\sum_i Q_i = Q$.

Flavor charges:

$$Q_\sigma(t) ::= G^{-1}_\theta(t) : Q_i : G_\theta(t) = \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^r_{\mathbf{k},\sigma}(t) - \beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\beta^r_{-\mathbf{k},\sigma}(t) \right)$$

CP violation and $SU(3)$

Modified Gell-Mann matrices:

$$\tilde{\lambda}_1 = \begin{pmatrix} 0 & e^{i\delta_2} & 0 \\ e^{-i\delta_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tilde{\lambda}_2 = \begin{pmatrix} 0 & -ie^{i\delta_2} & 0 \\ ie^{-i\delta_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tilde{\lambda}_4 = \begin{pmatrix} 0 & 0 & e^{-i\delta_5} \\ 0 & 0 & 0 \\ e^{i\delta_5} & 0 & 0 \end{pmatrix}$$

$$\tilde{\lambda}_5 = \begin{pmatrix} 0 & 0 & -ie^{-i\delta_5} \\ 0 & 0 & 0 \\ ie^{i\delta_5} & 0 & 0 \end{pmatrix}, \tilde{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i\delta_7} \\ 0 & e^{-i\delta_7} & 0 \end{pmatrix}, \tilde{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -ie^{i\delta_7} \\ 0 & ie^{-i\delta_7} & 0 \end{pmatrix}$$

$$\tilde{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \tilde{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Entanglement in neutrino oscillations.

- Flavor mixing and entanglement;
- Entanglement in neutrino oscillations:
  - Flavor entanglement;
  - Decoherence;
- Neutrino oscillations as a resource for quantum information.
- Flavor entanglement in Quantum Field Theory.

Entanglement in particle mixing

– Flavor mixing (neutrinos)

$$|\nu_e\rangle = \quad \cos\theta\,|\nu_1\rangle\, + \,\sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle\, + \,\cos\theta\,|\nu_2\rangle$$

• Correspondence with two-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2 \equiv |10\rangle, \qquad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2 \equiv |01\rangle,$$

where $|\rangle_i$ denotes states in the Hilbert space for neutrinos with mass m_i.

⇒ flavor states are entangled superpositions of the mass eigenstates:

$$|\nu_e\rangle = \quad \cos\theta\,|10\rangle\, + \,\sin\theta\,|01\rangle.$$

Single-particle entanglement[§]

– A state like $|\psi\rangle_{A,B} = |0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B$ is entangled;

– entanglement among field modes, rather than particles;

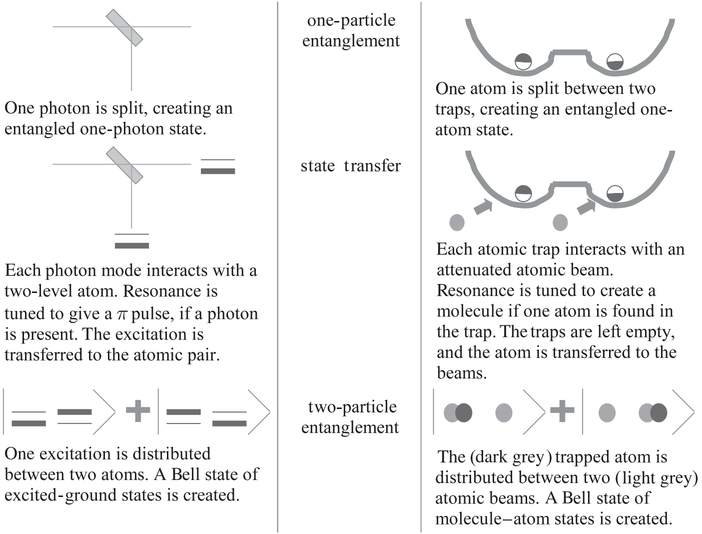
– entanglement is a property of composite systems, rather than of many-particle systems;

– entanglement and non-locality are not synonyms;

– single-particle entanglement is as good as two-particle entanglement for applications (quantum cryptography, teleportation, violation of Bell inequalities, etc..).

[§] J.van Enk, Phys. Rev. A (2005), (2006);
M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007);
J.A.Dunningham and V.Vedral, Phys. Rev. Lett. (2007).
S.B.Papp et al. Science (2009)
D.Salart et al. Phys. Rev. Lett (2010)
G.Björk, A.Laghaout, U.L.Andersen Phys. Rev. A (2012)

Protocols for extraction of single-particle entanglement (from M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007)



Multipartite entanglement in neutrino mixing[¶]

– Neutrino mixing (three flavors):

$$|\underline{\nu}_f\rangle \quad = \quad U(\tilde{\theta},\delta)\,|\underline{\nu}_m\rangle$$

with $|\underline{\nu}_f\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T$ and $|\underline{\nu}_m\rangle = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T$.

– Mixing matrix (MNSP)

$$U(\tilde{\theta},\delta) \quad = \quad \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

where $(\tilde{\theta},\delta) \equiv (\theta_{12}, \theta_{13}, \theta_{23}; \delta)$, $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$.

- Correspondence with three-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2|0\rangle_3 \equiv |100\rangle, \quad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2|0\rangle_3 \equiv |010\rangle,$$

$$|\nu_3\rangle \equiv |0\rangle_1|0\rangle_2|1\rangle_3 \equiv |001\rangle$$

[¶] M.B., F.Dell'Anno, S.De Siena, M.Di Mauro and F.Illuminati, Phys. Rev. D (2008).

Multipartite entanglement

– Characterization of entanglement for multipartite systems is a non-trivial task. Several approaches have been developed: global entanglement, tangle, geometric measures[‖], etc...

In the 3-qubit case, the two fundamental classes[**] of states are those of the GHZ state and of the W state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\,,$$

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)\,.$$

[‖] T.C.Wei and P.M.Goldbart Phys. Rev. A (2003);
M.B., F.Dell'Anno, S.De Siena and F.Illuminati, Phys. Rev. A (2008).
[**] W.Dür, G.Vidal, and J.I.Cirac, Phys. Rev. A (2000)

(Flavor) Entanglement in neutrino oscillations[††]

– Two-flavor neutrino states

$$|\underline{\nu}^{(f)}\rangle \;=\; \mathbf{U}(\tilde{\theta},\delta)\,|\underline{\nu}^{(m)}\rangle$$

where $|\underline{\nu}^{(f)}\rangle \;=\; (|\nu_e\rangle, |\nu_\mu\rangle)^T$ and $|\underline{\nu}^{(m)}\rangle \;=\; (|\nu_1\rangle, |\nu_2\rangle)^T$ and $\mathbf{U}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

– Flavor states at time t:

$$|\underline{\nu}^{(f)}(t)\rangle \;=\; \mathbf{U}(\tilde{\theta},\delta)\,\mathbf{U}_0(t)\,\mathbf{U}(\tilde{\theta},\delta)^{-1}\,|\underline{\nu}^{(f)}\rangle \equiv\; \widetilde{\mathbf{U}}(t)|\underline{\nu}^{(f)}\rangle\,,$$

with $\mathbf{U}_0(t) = \begin{pmatrix} e^{-iE_1 t} & 0 \\ 0 & e^{-iE_2 t} \end{pmatrix}$.

[††]M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2009).

– Transition probability for $\nu_\alpha \to \nu_\beta$

$$P_{\nu_\alpha \to \nu_\beta}(t) \,=\, |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 \,=\, |\widetilde{\mathbf{U}}_{\alpha\beta}(t)|^2 \,.$$

• We now take the flavor states at initial time as our qubits:

$$|\nu_e\rangle \equiv |1\rangle_e |0\rangle_\mu \equiv |10\rangle_f, \quad |\nu_\mu\rangle \equiv |0\rangle_e |1\rangle_\mu \equiv |01\rangle_f,$$

– Starting from $|10\rangle_f$ or $|01\rangle_f$, time evolution generates the (entangled) Bell-like states:

$$|\nu_\alpha(t)\rangle \,=\, \widetilde{\mathbf{U}}_{\alpha e}(t) |1\rangle_e |0\rangle_\mu + \widetilde{\mathbf{U}}_{\alpha \mu}(t) |0\rangle_e |1\rangle_\mu, \quad \alpha = e, \mu.$$

Entanglement measure

– Let $\rho = |\psi\rangle\langle\psi|$ be the density operator for a pure state $|\psi\rangle$

Bipartition of the N-partite system $S = \{S_1, S_2, \ldots, S_N\}$ in two subsystems:

$$S_{A_n} = \{S_{i_1}, S_{i_2}, \ldots, S_{i_n}\}, \quad 1 \leq i_1 < i_2 < \ldots < i_n \leq N; (1 \leq n < N)$$

and

$$S_{B_{N-n}} = \{S_{j_1}, S_{j_2}, \ldots, S_{j_{N-n}}\} \quad , 1 \leq j_1 < j_2 < \ldots < j_{N-n} \leq N; i_q \neq j_p$$

– Reduced density matrix of S_{A_n} after tracing over $S_{B_{N-n}}$:

$$\rho_{A_n} \equiv \rho_{i_1,i_2,\ldots,i_n} = Tr_{B_{N-n}}[\rho] = Tr_{j_1,j_2,\ldots,j_{N-n}}[\rho]$$

• Linear entropy associated to such a bipartition:

$$S_L^{(A_n;B_{N-n})}(\rho) = \frac{d}{d-1}(1 - Tr_{A_n}[\rho_{A_n}^2]),$$

d is the Hilbert-space dimension:
$d = \min\{\dim S_{A_n}, \dim S_{B_{N-n}}\} = \min\{2^n, 2^{N-n}\}$.

• Average linear entropy (global entanglement):

$$\langle S_L^{(n:N-n)}(\rho)\rangle = \begin{pmatrix} N \\ n \end{pmatrix}^{-1} \sum_{A_n} S_L^{(A_n;B_{N-n})}(\rho),$$

sum over all the possible bi-partitions of the system in two subsystems, respectively with n and $N-n$ elements ($1 \leq n < N$).

Entanglement in neutrino oscillations: two-flavors

Consider the density matrix for the electron neutrino state $\rho^{(e)} = |\nu_e(t)\rangle\langle\nu_e(t)|$, and trace over mode $\mu \;\Rightarrow\; \rho_e^{(e)}$.

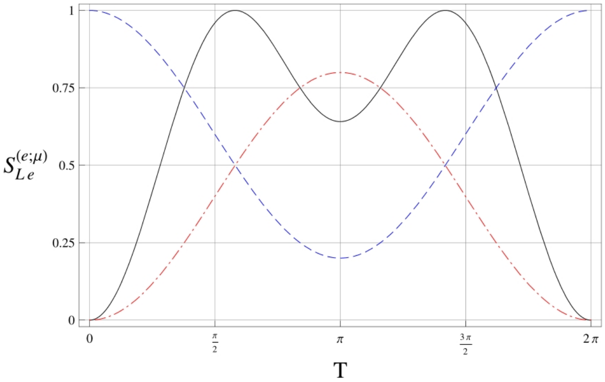
– The associated linear entropy is :

$$S_L^{(e;\mu)}(\rho^{(e)}) \;=\; 4\,|\widetilde{\mathbf{U}}_{e\mu}(t)|^2\,|\widetilde{\mathbf{U}}_{ee}(t)|^2 \;=\; 4\,P_{\nu_e\to\nu_e}(t)\,P_{\nu_e\to\nu_\mu}(t)$$

– The linear entropy for the state $\rho^{(\alpha)}$ is:

$$\begin{aligned} S_{L\alpha}^{(e;\mu)} = S_{L\alpha}^{(\mu;e)} = \langle S_{L\alpha}^{(1:1)}\rangle &= 4\,|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2\,|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2 \\ &= 4|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2\,(1-|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2) \\ &= 4|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2\,(1-|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2)\,. \end{aligned}$$

• Linear entropy given by product of transition probabilities !



Linear entropy $S_{Le}^{(e;\mu)}$ (full) as a function of the scaled time $T = \frac{2Et}{\Delta m_{12}^2}$, with $\sin^2\theta = 0.314$. Transition probabilities $P_{\nu_e \to \nu_e}$ (dashed) and $P_{\nu_e \to \nu_\mu}$ (dot-dashed) are reported for comparison.

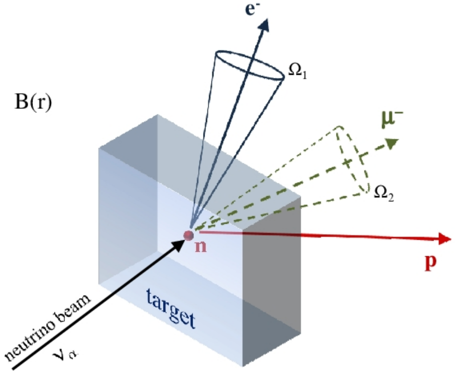
Neutrino oscillations as a resource for quantum information

• Single-particle entanglement encoded in flavor states $|\underline{\nu}^{(f)}(t)\rangle$ is a real physical resource that can be used, at least in principle, for protocols of quantum information.

– Experimental scheme for the transfer of the flavor entanglement of a neutrino beam into a single-particle system with *spatially separated modes.*

Charged-current interaction between a neutrino ν_α with flavor α and a nucleon N gives a lepton α^- and a baryon X:

$$\nu_\alpha + N \longrightarrow \alpha^- + X \,.$$



Generation of a single-particle entangled lepton state (two flavors):

In the target the charged-current interaction occurs: $\nu_\alpha + n \longrightarrow \alpha^- + p$ with $\alpha = e\,,\mu$.

A spatially nonuniform magnetic field $\mathbf{B}(\mathbf{r})$ constraints the momentum of the outgoing lepton within a solid angle Ω_i, and ensures spatial separation between lepton paths.

The reaction produces a superposition of electronic and muonic spatially separated states.

• Given the initial Bell-like superposition $|\nu_\alpha(t)\rangle$ the unitary process associated with the weak interaction leads to the superposition

$$|\alpha(t)\rangle \;=\; \Lambda_e|1\rangle_e|0\rangle_\mu + \Lambda_\mu|0\rangle_e|1\rangle_\mu\,,$$

where $|\Lambda_e|^2 + |\Lambda_\mu|^2 = 1$, and $|k\rangle_\alpha$, with $k = 0, 1$, represents the lepton qubit.

The coefficients Λ_α are proportional to $\widetilde{\mathbf{U}}_{\alpha\beta}(t)$ and to the cross sections associated with the creation of an electron or a muon.

• Analogy with single-photon system: quantum uncertainty on the so-called "*which path*" of the photon at the output of an unbalanced beam splitter ⇔ uncertainty on the "*which flavor*" of the produced lepton.

The coefficients Λ_α plays the role of the transmissivity and of the reflectivity of the beam splitter.