Anisotropic dissipative fluid dynamics – theory and applications in heavy-ion physics

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Microscopic foundations of ideal fluid dynamics

Ideal fluid dynamics: fluid is in local thermodynamical equilibrium

$\Longrightarrow$ **single-particle distribution function:** $\boxed{f_{0\mathrm{k}} = \left[\exp\left(-\alpha_0 + \beta_0 E_{\mathrm{k}u}\right) + a\right]^{-1}}$

where: $\beta_0 = 1/T$, T temperature, $\alpha_0 = \beta_0 \mu$, μ chemical potential,
$E_{\mathrm{k}u} = k^\mu u_\mu$, with k^μ particle 4-momentum, $u^\mu = \gamma(1, \vec{v})$ fluid 4-velocity,
$a = \pm 1, 0$ for fermions/bosons, Boltzmann particles

Boltzmann equation: $\boxed{k^\mu \partial_\mu f_{\mathrm{k}} = C[f]}$

$\Longrightarrow$ **0^{th} and 1^{st} moment of the Boltzmann equation:**

$$\boxed{\begin{aligned} \partial_\mu N^\mu &= \mathcal{C} \\ \partial_\mu T^{\mu\nu} &= \mathcal{C}^\nu \end{aligned}}$$

where: $N^\mu \equiv \int_k k^\mu f_{\mathrm{k}}$ particle no. 4-current,
$T^{\mu\nu} \equiv \int_k k^\mu k^\nu f_{\mathrm{k}}$ energy-momentum tensor,
$\int_k \equiv g \int \frac{d^3k}{(2\pi)^3 k_0}$, g: internal quantum no. degeneracy of momentum state

Note: $\mathcal{C} \equiv \int_k C[f] = 0$ and $\mathcal{C}^\nu \equiv \int_k k^\nu C[f] \equiv 0$ for binary elastic collisions
(particle no. and 4-momenta are microscopic collisional invariants)

$\Longrightarrow$ **macroscopic conservation of particle no., energy, and momentum!**

$\Longrightarrow$ set $f_{\mathrm{k}} \equiv f_{0\mathrm{k}}$ (**Note:** $f_{0\mathrm{k}}$ is **not** a solution of the Boltzmann equation!)

$\Longrightarrow$ equations of motion closed – 5 eqs., 5 unknowns: α_0, β_0, $u^\mu(3)$

Microscopic foundations of dissipative fluid dynamics (I)

equations of motion no longer closed:

$\Longrightarrow$ **in Landau frame, where u^μ follows flow of energy**

$$\boxed{\begin{aligned} \partial_\mu N^\mu &= 0 \\ \partial_\mu T^{\mu\nu} &= 0 \end{aligned}} \quad \Longleftrightarrow \quad \boxed{\begin{aligned} &\dot{n} + n\,\theta + \partial \cdot n = 0 \\ &\dot{\epsilon} + (\epsilon + p + \Pi)\,\theta - \pi^{\mu\nu}\,\partial_\mu u_\nu = 0 \\ &(\epsilon + p)\dot{u}^\mu = \nabla^\mu(p + \Pi) - \Pi\dot{u}^\mu - \Delta^{\mu\nu}\,\partial^\lambda \pi_{\nu\lambda} \end{aligned}}$$

where: $\dot{A} \equiv u^\mu \partial_\mu A$ **comoving derivative,**
$\theta \equiv \partial_\mu u^\mu$ **expansion scalar,**
$\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ **3-space projector onto direction orthogonal to u^μ,**
$\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$ **3-space gradient orthogonal to u^μ**

and: n **particle density,**
ϵ **energy density,**
p **pressure,**
Π **bulk viscous pressure,**
n^μ **particle diffusion current,**
$\pi^{\mu\nu}$ **shear-stress tensor**

$\Longrightarrow$ **need additional equations of motion for Π, n^μ, $\pi^{\mu\nu}$!**

Microscopic foundations of dissipative fluid dynamics (II)

Consider small deviations from local thermodynamical equilibrium:

$$\boxed{f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}}} \qquad |\delta f_{\mathbf{k}}| \ll 1$$

$\Longrightarrow$ **irreducible moments of $\delta f_{\mathbf{k}}$:** $\boxed{\rho_r^{\mu_1 \cdots \mu_\ell} \equiv \int_k E_{\mathbf{k}u}^r \, k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle} \, \delta f_{\mathbf{k}}}$

where: $A^{\langle \mu_1 \cdots \mu_\ell \rangle} \equiv \Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} A^{\nu_1 \cdots \nu_\ell}$,
$\Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell}$ projectors onto subspaces orthogonal to u^μ , formed from $\Delta^{\mu\nu}$,
symmetric in μ_i, ν_j , traceless,

Note: $-\frac{m^2}{3}\rho_0 \equiv \Pi\,,\;\; \rho_0^\mu \equiv n^\mu\,,\; \rho_0^{\mu\nu} \equiv \pi^{\mu\nu}\,,$
matching conditions in Landau frame: $\rho_1 = \rho_2 = \rho_1^\mu = 0$

$\Longrightarrow$ **derive equations of motion for irreducible moments:**

$$\boxed{\dot{\rho}_r^{\langle \mu_1 \cdots \mu_\ell \rangle} \equiv \Delta^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} \, u^\alpha \partial_\alpha \int_k E_{\mathbf{k}u}^r \, k^{\langle \nu_1} \cdots k^{\nu_\ell \rangle} \delta f_{\mathbf{k}}}$$

$\Longrightarrow$ **use Boltzmann equation:** $\boxed{\delta \dot{f}_{\mathbf{k}} = -\dot{f}_{0\mathbf{k}} - \frac{1}{E_{\mathbf{k}u}} \left\{ k^\mu \nabla_\mu \left(f_{0\mathbf{k}} + \delta f_{\mathbf{k}} \right) - C[f] \right\}}$

$\Longrightarrow$ **system of infinitely many coupled equations for irreducible moments $\rho_r^{\mu_1 \cdots \mu_\ell}$, completely equivalent to Boltzmann equation $\Longrightarrow$ truncation required!**

Microscopic foundations of dissipative fluid dynamics (III)

systematic power counting:

$$\boxed{\begin{array}{ll} \mathrm{Kn} \equiv \dfrac{\ell_{\mathrm{mfp}}}{L_{fluid}} \sim \ell_{\mathrm{mfp}}\, \partial_\mu & \text{Knudsen number} \\ \mathrm{Re}^{-1} \equiv \dfrac{\Pi}{p} \sim \dfrac{n^\mu}{n} \sim \dfrac{\pi^{\mu\nu}}{p} & \text{inverse Reynolds number} \end{array}}$$

with pressure p, particle density n

$\Longrightarrow$ for $\ell \geq 3 : \rho_r^{\mu_1 \cdots \mu_\ell} \sim O(\mathrm{Kn}^2, \mathrm{Kn}\,\mathrm{Re}^{-1})$ $\Longrightarrow$ will be neglected

$\Longrightarrow$ linearize collision integral: $\int_k E_{\mathrm{k}u}^{r-1}\, k^{\langle \mu_1} \cdots k^{\mu_\ell \rangle}\, C[f] = - \sum_{n=0}^{N_\ell} \mathcal{A}_{rn}^{(\ell)}\, \rho_n^{\mu_1 \cdots \mu_\ell} + O(\delta f_{\mathrm{k}}^2)$

$\Longrightarrow$ linearized equations of motion for irreducible moments:

$$\boxed{\begin{aligned} \dot{\vec{\rho}} + \mathcal{A}^{(0)}\, \vec{\rho} &= \vec{\alpha}^{(0)} \theta + O(\rho \times \mathrm{Kn}) \\ \dot{\vec{\rho}}^{\langle \mu \rangle} + \mathcal{A}^{(1)}\, \vec{\rho}^{\,\mu} &= \vec{\alpha}^{(1)} \nabla^\mu \alpha + O(\rho \times \mathrm{Kn}) \\ \dot{\vec{\rho}}^{\langle \mu\nu \rangle} + \mathcal{A}^{(2)}\, \vec{\rho}^{\,\mu\nu} &= 2\, \vec{\alpha}^{(2)} \sigma^{\mu\nu} + O(\rho \times \mathrm{Kn}) \end{aligned}}$$

where $\sigma^{\mu\nu} \equiv \nabla^{\langle \mu} u^{\nu \rangle}$

$\Longrightarrow$ diagonalize collision matrix: $(\Omega^{-1})^{(\ell)} \mathcal{A}^{(\ell)} \Omega^{(\ell)} = \mathrm{diag}(\chi_0^{(\ell)}, \ldots, \chi_i^{(\ell)}, \ldots) \equiv \chi^{(\ell)}$

$\Longrightarrow$ equations of motion for eigenmodes $\vec{X}^{\mu_1 \cdots \mu_\ell} = (\Omega^{-1})^{(\ell)} \vec{\rho}^{\,\mu_1 \cdots \mu_\ell}$ decouple:

$$\boxed{\begin{aligned} \dot{\vec{X}} + \chi^{(0)} \vec{X} &= \vec{\beta}^{(0)} \theta + O(X \times \mathrm{Kn}) \\ \dot{\vec{X}}^{\langle \mu \rangle} + \chi^{(1)} \vec{X}^\mu &= \vec{\beta}^{(1)} \nabla^\mu \alpha + O(X \times \mathrm{Kn}) \\ \dot{\vec{X}}^{\langle \mu\nu \rangle} + \chi^{(2)} \vec{X}^{\mu\nu} &= \vec{\beta}^{(2)} \sigma^{\mu\nu} + O(X \times \mathrm{Kn}) \end{aligned}}$$

where $\vec{\beta}^{(\ell)} = \left(\Omega^{-1}\right)^{(\ell)} \vec{\alpha}^{(\ell)}$

Microscopic foundations of dissipative fluid dynamics (IV)

$\Longrightarrow$ slowest eigenmodes (w/o r.o.g. X_0, X_0^μ, $X_0^{\mu\nu}$) remain dynamical, faster ones ($i \neq 0$) are replaced by their asymptotic values:

$$X_i \simeq \frac{\beta_i^{(0)}}{\chi_i^{(0)}}\,\theta\,, \quad X_i^\mu \simeq \frac{\beta_i^{(1)}}{\chi_i^{(1)}}\,\nabla^\mu\alpha\,, \quad X_i^{\mu\nu} \simeq \frac{\beta_i^{(2)}}{\chi_i^{(2)}}\,\sigma^{\mu\nu}$$

Note: systematic improvement possible by making faster eigenmodes dynamical

G.S. Denicol, H. Niemi, I. Bouras, E. Molnar, Z. Xu, DHR, C. Greiner, PRD 89 (2014) 7, 074005

$\Longrightarrow$ since $\vec{\rho}^{\,\mu_1\cdots\mu_\ell} = \Omega^{(\ell)}\,\vec{X}^{\mu_1\cdots\mu_\ell}$:

$$\rho_i \simeq \Omega_{i0}^{(0)} X_0 + \sum_{j=3}^{N_0} \Omega_{ij}^{(0)} \frac{\beta_j^{(0)}}{\chi_j^{(0)}}\,\theta$$

$$\rho_i^\mu \simeq \Omega_{i0}^{(1)} X_0^\mu + \sum_{j=2}^{N_1} \Omega_{ij}^{(1)} \frac{\beta_j^{(1)}}{\chi_j^{(1)}}\,\nabla^\mu\alpha$$

$$\rho_i^{\mu\nu} \simeq \Omega_{i0}^{(2)} X_0^{\mu\nu} + \sum_{j=1}^{N_2} \Omega_{ij}^{(2)} \frac{\beta_i^{(2)}}{\chi_j^{(2)}}\,\sigma^{\mu\nu}$$

$\Longrightarrow$ for $i = 0$: express X_0, X_0^μ, $X_0^{\mu\nu}$ in terms of Π, n^μ, $\pi^{\mu\nu}$ as well as θ, $\nabla^\mu\alpha$, $\sigma^{\mu\nu}$

$\Longrightarrow$ reinsert back, express ρ_i, ρ_i^μ, $\rho_i^{\mu\nu}$ in terms of Π, n^μ, $\pi^{\mu\nu}$ as well as θ, $\nabla^\mu\alpha$, $\sigma^{\mu\nu}$:

$$\boxed{\begin{aligned} \frac{m^2}{3}\,\rho_i &\simeq -\Omega_{i0}^{(0)}\Pi + \left(\zeta_i - \Omega_{i0}^{(0)}\zeta_0\right)\theta \\ \rho_i^\mu &\simeq \Omega_{i0}^{(1)} n^\mu + \left(\kappa_i - \Omega_{i0}^{(1)}\kappa_0\right)\nabla^\mu\alpha \\ \rho_i^{\mu\nu} &\simeq \Omega_{i0}^{(2)}\pi^{\mu\nu} + 2\left(\eta_i - \Omega_{i0}^{(2)}\eta_0\right)\sigma^{\mu\nu} \end{aligned}}$$

where $\zeta_i = \frac{m^2}{3}\sum_{r=0,\neq 1,2}^{N_0} \tau_{ir}^{(0)}\alpha_r^{(0)}\,,\ \kappa_i = \sum_{r=0,\neq 1}^{N_1} \tau_{ir}^{(1)}\alpha_r^{(1)}\,,\ \eta_i = \sum_{r=0}^{N_2} \tau_{ir}^{(2)}\alpha_r^{(2)}\,,\ \tau^{(\ell)} = \Omega^{(\ell)}(\chi^{-1})^{(\ell)}(\Omega^{-1})^{(\ell)}$

Microscopic foundations of dissipative fluid dynamics (V)

$\Longrightarrow$ **equations of motion for Π, n^μ, $\pi^{\mu\nu}$:**

$$\boxed{\begin{aligned} \tau_\Pi \dot{\Pi} + \Pi &= -\zeta_0 \theta + \mathcal{K} + \mathcal{J} + \mathcal{R} \\ \tau_n \dot{n}^{<\mu>} + n^\mu &= \kappa_0 \nabla^\mu \alpha + \mathcal{K}^\mu + \mathcal{J}^\mu + \mathcal{R}^\mu \\ \tau_\pi \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} &= 2\,\eta_0\, \sigma^{\mu\nu} + \mathcal{K}^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu} \end{aligned}}$$

Kn^2:

$$\mathcal{K} = \bar{\zeta}_1\, \omega_{\mu\nu}\, \omega^{\mu\nu} + \bar{\zeta}_2\, \sigma^{\mu\nu}\, \sigma_{\mu\nu} + \bar{\zeta}_3\, \theta^2 + \bar{\zeta}_4\, (\nabla\alpha)^2 + \bar{\zeta}_5\, (\nabla p)^2 + \bar{\zeta}_6\, \nabla_\mu \alpha \nabla^\mu p + \bar{\zeta}_7\, \nabla^2 \alpha + \bar{\zeta}_8\, \nabla^2 p\,,$$

$$\mathcal{K}^\mu = \bar{\kappa}_1\, \sigma^{\mu\nu}\, \nabla_\nu \alpha + \bar{\kappa}_2\, \sigma^{\mu\nu}\, \nabla_\nu p + \bar{\kappa}_3\, \theta\, \nabla^\mu \alpha + \bar{\kappa}_4\, \theta\, \nabla^\mu p + \bar{\kappa}_5\, \omega^{\mu\nu}\, \nabla_\nu \alpha + \bar{\kappa}_6\, \Delta^{\mu\lambda} \partial^\nu \sigma_{\lambda\nu} + \bar{\kappa}_7\, \nabla^\mu \theta\,,$$

$$\begin{aligned} \mathcal{K}^{\mu\nu} = {} & \bar{\eta}_1\, \omega_\lambda{}^{\langle\mu}\, \omega^{\nu\rangle\lambda} + \bar{\eta}_2\, \theta\, \sigma^{\mu\nu} + \bar{\eta}_3\, \sigma_\lambda{}^{\langle\mu}\, \sigma^{\nu\rangle\lambda} + \bar{\eta}_4\, \sigma_\lambda{}^{\langle\mu}\, \omega^{\nu\rangle\lambda} + \bar{\eta}_5\, \nabla^{\langle\mu}\alpha\, \nabla^{\nu\rangle}\alpha \\ & + \bar{\eta}_6\, \nabla^{\langle\mu} p\, \nabla^{\nu\rangle} p + \bar{\eta}_7\, \nabla^{\langle\mu}\alpha\, \nabla^{\nu\rangle} p + \bar{\eta}_8\, \nabla^{\langle\mu}\nabla^{\nu\rangle}\alpha + \bar{\eta}_9\, \nabla^{\langle\mu}\nabla^{\nu\rangle} p \end{aligned}$$

Re^{-1}Kn:

$$\mathcal{J} = -\ell_{\Pi n}\, \nabla_\mu n^\mu - \tau_{\Pi n}\, n^\mu \nabla_\mu p - \delta_{\Pi\Pi}\, \theta\, \Pi - \lambda_{\Pi n}\, n^\mu \nabla_\mu \alpha + \lambda_{\Pi\pi}\, \pi^{\mu\nu} \sigma_{\mu\nu}$$

$$\begin{aligned} \mathcal{J}^\mu = {} & \tau_n\, \omega^{\mu\nu}\, n_\nu - \delta_{nn}\, \theta\, n^\mu - \ell_{n\Pi}\, \nabla^\mu \Pi + \ell_{n\pi} \Delta^{\mu\nu}\, \nabla^\lambda \pi_{\nu\lambda} + \tau_{n\Pi}\, \Pi\, \nabla^\mu p - \tau_{n\pi}\, \pi^{\mu\nu}\, \nabla_\nu p - \lambda_{nn}\, \sigma^{\mu\nu}\, n_\nu \\ & + \lambda_{n\Pi}\, \Pi\, \nabla^\mu \alpha - \lambda_{n\pi}\, \pi^{\mu\nu}\, \nabla_\nu \alpha \end{aligned}$$

$$\begin{aligned} \mathcal{J}^{\mu\nu} = {} & 2\,\tau_\pi\, \pi_\lambda{}^{\langle\mu}\, \omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\, \theta\, \pi^{\mu\nu} - \tau_{\pi\pi}\, \pi_\lambda{}^{\langle\mu}\, \sigma^{\nu\rangle\lambda} + \lambda_{\pi\Pi}\, \Pi\, \sigma^{\mu\nu} - \tau_{\pi n}\, n^{\langle\mu}\, \nabla^{\nu\rangle} p + \ell_{\pi n}\, \nabla^{\langle\mu} n^{\nu\rangle} \\ & + \lambda_{\pi n}\, n^{\langle\mu} \nabla^{\nu\rangle} \alpha \end{aligned}$$

where $\omega^{\mu\nu} \equiv \left(\nabla^\mu u^\nu - \nabla^\nu u^\mu\right)/2$

Re^{-2}:

$$\mathcal{R} = \varphi_1\, \Pi^2 + \varphi_2\, n_\mu n^\mu + \varphi_3\, \pi^{\mu\nu} \pi_{\mu\nu}$$

$$\mathcal{R}^\mu = \varphi_4\, \pi^{\mu\nu}\, n_\nu + \varphi_5\, \Pi\, n^\mu$$

$$\mathcal{R}^{\mu\nu} = \varphi_6\, \Pi\, \pi^{\mu\nu} + \varphi_7\, \pi_\lambda{}^{\langle\mu}\, \pi^{\nu\rangle\lambda} + \varphi_8\, n^{\langle\mu}\, n^{\nu\rangle}$$

G.S. Denicol, H. Niemi, E. Molnar, DHR,
PRD 85 (2012) 114047,
Erratum PRD 91 (2015) 3, 039902

Microscopic foundations of dissipative fluid dynamics (VI)

Single-particle distribution function:

$$\boxed{f_{\mathbf{k}} = f_{0\mathbf{k}} \left[1 + (1 - a f_{0\mathbf{k}}) \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \mathcal{H}_{\mathbf{k}n}^{(\ell)} \rho_n^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \right]}$$

where $\mathcal{H}_{\mathbf{k}n}^{(\ell)} = \frac{W^{(\ell)}}{\ell!} \sum_{m=n}^{N_\ell} a_{mn}^{(\ell)} P_{\mathbf{k}m}^{(\ell)}$, with $P_{\mathbf{k}n}^{(\ell)} = \sum_{r=0}^{n} a_{nr}^{(\ell)} E_{\mathbf{k}u}^{r}$ polynomials of order n in $E_{\mathbf{k}u}$,

with coefficients $a_{nr}^{(\ell)}$ determined such that $\frac{W^{(\ell)}}{(2\ell+1)!!} \int_k \left(\Delta^{\alpha\beta} k_\alpha k_\beta \right)^\ell P_{\mathbf{k}n}^{(\ell)} P_{\mathbf{k}m}^{(\ell)} f_{0\mathbf{k}} (1 - a f_{0\mathbf{k}}) = \delta_{mn}$

$\Longrightarrow$ explicitly for $\ell \leq 2$:

$$\begin{aligned} \delta f_{\mathbf{k}} = f_{0\mathbf{k}} (1 - a f_{0\mathbf{k}}) \Bigg(& -\frac{3}{m^2} \left\{ \mathcal{H}_{\mathbf{k}0}^{(0)} \Pi + \sum_{n=3}^{N_0} \mathcal{H}_{\mathbf{k}n}^{(0)} \left[-\Omega_{n0}^{(0)} \Pi + \left(\zeta_n - \Omega_{n0}^{(0)} \zeta_0 \right) \theta \right] \right\} \\ & + \mathcal{H}_{\mathbf{k}0}^{(1)} n^\mu k_\mu + \sum_{n=2}^{N_1} \mathcal{H}_{\mathbf{k}n}^{(1)} \left[\Omega_{n0}^{(1)} n^\mu + \left(\kappa_n - \Omega_{n0}^{(1)} \kappa_0 \right) \nabla^\mu \alpha \right] k_\mu \\ & + \mathcal{H}_{\mathbf{k}0}^{(2)} \pi^{\mu\nu} k_\mu k_\nu + \sum_{n=1}^{N_2} \mathcal{H}_{\mathbf{k}n}^{(2)} \left[\Omega_{n0}^{(2)} \pi^{\mu\nu} + 2 \left(\eta_n - \Omega_{n0}^{(2)} \eta_0 \right) \sigma^{\mu\nu} \right] k_\mu k_\nu \Bigg) \end{aligned}$$

$$\mathcal{H}_{\mathbf{k}0}^{(2)} = \frac{1}{2\, J_{42}} \left(1 + \sum_{m=1}^{N_2} \sum_{r=0}^{m} a_{m0}^{(2)} a_{mr}^{(2)} E_{\mathbf{k}u}^{r} \right)$$

usually: $\delta f_{\mathbf{k}} = f_{0\mathbf{k}} (1 - a f_{0\mathbf{k}}) \frac{1}{2T^2(\epsilon + p)} \pi^{\mu\nu} k_\mu k_\nu$ with energy density ϵ

Anisotropic fluid dynamics

Initial gradients in heavy-ion collisions are large

$\Longrightarrow$ **deviations from local thermodynamical equilibrium are large!**

$\Longrightarrow$ **may invalidate dissipative fluid dynamics**

Idea: "resum" dissipative corrections into single-particle distribution function,

e.g.: W. Florkowski, PLB 668 (2008) 32; M. Martinez, M. Strickland, PRC 81 (2010) 024906

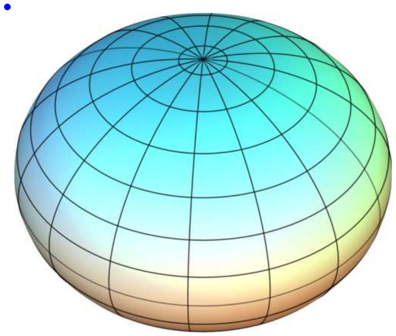
$$\boxed{\hat{f}_{0\mathbf{k}} = \left[\exp\left(-\hat{\alpha} + \hat{\beta}_u \sqrt{E_{\mathbf{k}u}^2 + \xi\, E_{\mathbf{k}l}^2}\right) + a\right]^{-1}}$$

where $E_{\mathbf{k}l} \equiv -l^\mu k_\mu$ **, with** l^μ **direction of anisotropy,** $l^\mu l_\mu = -1$, $l^\mu u_\mu = 0$,

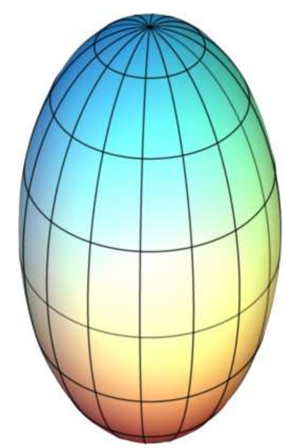
usually: $l^\mu = \gamma_z(v_z, 0, 0, 1)$, $\gamma_z = (1 - v_z^2)^{-1/2}$,

ξ **anisotropy parameter**

$\Longrightarrow$ **in LR frame of fluid:**



$\xi > 0$



$\xi < 0$

$\Longrightarrow$ **5 conservation equations determine** $\hat{\alpha}$, $\hat{\beta}_u$, u^μ **(3)**

$\Longrightarrow$ **need additional equation to determine** ξ**!**

Microscopic foundations of anisotropic dissipative fluid dynamics (I)

$$\boxed{f_{\mathrm{k}} = f_{0\mathrm{k}} + \delta f_{\mathrm{k}} \equiv \hat{f}_{0\mathrm{k}} + \delta \hat{f}_{\mathrm{k}}}$$

If $|\delta f_{\mathrm{k}}| \sim 1$, choose $\hat{f}_{0\mathrm{k}}$ such that $|\delta \hat{f}_{\mathrm{k}}| \ll 1$

$\Longrightarrow$ **improved convergence properties of expansion around $\hat{f}_{0\mathrm{k}}$!**

D. Bazow, U.W. Heinz, M. Strickland, PRC 90 (2014) 5, 054910

E. Molnár, H. Niemi, DHR, PRD 93 (2016) 11, 114025

$\Longrightarrow$ **irreducible moments of $\delta \hat{f}_{\mathrm{k}}$:** $\boxed{\hat{\rho}^{\mu_1 \cdots \mu_\ell}_{rs} \equiv \int_k E^r_{\mathrm{k}u}\, E^s_{\mathrm{k}l}\, k^{\{\mu_1} \cdots k^{\mu_\ell\}}\, \delta \hat{f}_{\mathrm{k}}}$

where: $A^{\{\mu_1 \cdots \mu_\ell\}} \equiv \Xi^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell} A^{\nu_1 \cdots \nu_\ell}$,

$\Xi^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell}$ **projectors onto subspaces orthogonal to both u^μ and l^μ , formed from $\Xi^{\mu\nu}$,**

symmetric in μ_i, ν_j , traceless,

$\Xi^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu + l^\mu l^\nu$ **2-space projector onto direction orthogonal to both u^μ and l^μ**

$\Longrightarrow$ **derive equations of motion for irreducible moments:**

$$\boxed{\dot{\hat{\rho}}^{\{\mu_1 \cdots \mu_\ell\}}_{rs} \equiv \Xi^{\mu_1 \cdots \mu_\ell}_{\nu_1 \cdots \nu_\ell}\, u^\alpha \partial_\alpha \int_k E^r_{\mathrm{k}u}\, E^s_{\mathrm{k}l}\, k^{\{\nu_1} \cdots k^{\nu_\ell\}} \delta \hat{f}_{\mathrm{k}}}$$

$\Longrightarrow$ **use Boltzmann equation:**

$$\boxed{\delta \dot{\hat{f}}_{\mathrm{k}} = -\dot{\hat{f}}_{0\mathrm{k}} - \frac{1}{E_{\mathrm{k}u}} \left\{ -E_{\mathrm{k}l} D_l \left(\hat{f}_{0\mathrm{k}} + \delta \hat{f}_{\mathrm{k}} \right) + k^\mu \tilde{\nabla}_\mu \left(\hat{f}_{0\mathrm{k}} + \delta \hat{f}_{\mathrm{k}} \right) - C[f] \right\}}$$

where: $D_l \equiv -l^\mu \partial_\mu$, $\tilde{\nabla}^\mu \equiv \Xi^{\mu\nu} \partial_\nu$

Microscopic foundations of anisotropic dissipative fluid dynamics (II)

Truncation: so far, no eigenmode analysis, only 14-moment approximation

Define
$$\boxed{\hat{I}_{nrq}(\hat{\alpha},\hat{\beta}_u,\xi) \equiv \frac{1}{(2q)!!}\int_k E^n_{\mathbf{k}u}\, E^r_{\mathbf{k}l}\, (-\Xi^{\alpha\beta}k_\alpha k_\beta)^q\, \hat{f}_{0\mathbf{k}}}$$

$\Longrightarrow$ **the 14 moments are:**

particle density	$n \equiv \hat{n} = \hat{I}_{100} \Longleftrightarrow \hat{\rho}_{10} = 0$ (1st Landau matching cond.)
particle diffusion in l^μ-direction	$n_l \equiv \hat{n}_l + \hat{\rho}_{01} = \hat{I}_{110} + \hat{\rho}_{01}$
energy density	$e \equiv \hat{e} = \hat{I}_{200} \Longleftrightarrow \hat{\rho}_{20} = 0$ (2nd Landau matching cond.)
heat flow in l^μ-direction	$M \equiv \hat{M} + \hat{\rho}_{11} = \hat{I}_{210} + \hat{\rho}_{11}$
pressure in l^μ-direction	$P_l \equiv \hat{P}_l = \hat{I}_{220} \Longleftrightarrow \hat{\rho}_{02} = 0$ (3rd Landau matching cond.)
transverse pressure	$P_\perp \equiv \hat{P}_\perp + \frac{3}{2}\Pi = \hat{I}_{201} - \frac{m_0^2}{2}\hat{\rho}_{00}$
particle diffusion in transverse direction	$V^\mu_\perp \equiv \hat{\rho}^\mu_{00}$
heat flow in transverse direction	$W^\mu_{\perp u} \equiv \hat{\rho}^\mu_{10}$
shear-stress current in l^μ-direction	$W^\mu_{\perp l} \equiv \hat{\rho}^\mu_{01}$
shear-stress tensor in transverse direction	$\pi^{\mu\nu}_\perp \equiv \hat{\rho}^{\mu\nu}_{00}$

$\Longrightarrow$ **Landau frame:** $M = W^\mu_{\perp u} = 0 \Longleftrightarrow \hat{\rho}_{11} = -\hat{M}\,,\; \hat{\rho}^\mu_{10} = 0$

$\Longrightarrow$ **eliminate all other moments by linear relation:**

$$\hat{\rho}^{\mu_1\cdots\mu_\ell}_{ij} = (-1)^\ell \ell! \sum_{n=0}^{N_\ell}\sum_{m=0}^{N_\ell - n} \hat{\rho}^{\mu_1\cdots\mu_\ell}_{nm}\gamma^{(\ell)}_{injm}$$ where $\gamma^{(\ell)}_{injm}$ function of $\hat{\alpha}$, $\hat{\beta}_u$, ξ

Note: for $\hat{f}_{0\mathbf{k}}(\xi)$: $\hat{n}_l = \hat{M} \equiv 0$!

Microscopic foundations of anisotropic dissipative fluid dynamics (III)

$\Longrightarrow$ **5 conservation equations:**

$$
\begin{aligned}
0 &= \dot{\hat{n}} + \hat{n}\left(l_\mu D_l u^\mu + \tilde{\theta}\right) - D_l n_l + n_l\left(\tilde{\theta}_l - l_\mu \dot{u}^\mu\right) - V_\perp^\mu\left(\dot{u}_\mu + D_l l_\mu\right) + \tilde{\nabla}_\mu V_\perp^\mu \\
0 &= \dot{\hat{e}} + \left(\hat{e} + \hat{P}_l\right) l_\mu D_l u^\mu + \left(\hat{e} + \hat{P}_\perp + \frac{3}{2}\Pi\right)\tilde{\theta} + W_{\perp l}^\mu\left(D_l u_\mu - l_\nu \tilde{\nabla}_\mu u^\nu\right) - \pi_\perp^{\mu\nu}\tilde{\sigma}_{\mu\nu} \\
0 &= \left(\hat{e} + \hat{P}_l\right) l_\mu \dot{u}^\mu + D_l \hat{P}_l + \left(\hat{P}_\perp - \hat{P}_l + \frac{3}{2}\Pi\right)\tilde{\theta}_l + W_{\perp l}^\mu\left(\dot{u}_\mu + 2\, D_l l_\mu + l_\nu \tilde{\nabla}_\mu u^\nu\right) - \tilde{\nabla}_\mu W_{\perp l}^\mu - \pi_\perp^{\mu\nu}\tilde{\sigma}_{l,\mu\nu} \\
0 &= \left(\hat{e} + \hat{P}_\perp + \frac{3}{2}\Pi\right)\Xi^\alpha_\nu \dot{u}^\nu - \tilde{\nabla}^\alpha\left(\hat{P}_\perp + \frac{3}{2}\Pi\right) + \left(\hat{P}_\perp - \hat{P}_l + \frac{3}{2}\Pi\right)\Xi^\alpha_\nu D_l l^\nu - \Xi^\alpha_\nu D_l W_{\perp l}^\nu + W_{\perp l}^\alpha\left(\frac{3}{2}\tilde{\theta}_l - l_\mu \dot{u}^\mu\right) \\
&+ W_{\perp l,\nu}\left(\tilde{\sigma}_l^{\alpha\nu} - \tilde{\omega}_l^{\alpha\nu}\right) - \pi_\perp^{\mu\alpha}\left(\dot{u}_\mu + D_l l_\mu\right) + \Xi^\alpha_\nu \tilde{\nabla}_\mu \pi_\perp^{\mu\nu}
\end{aligned}
$$

where $\tilde{\theta} \equiv \tilde{\nabla}_\mu u^\mu$, $\tilde{\theta}_l \equiv \tilde{\nabla}_\mu l^\mu$, $\tilde{\sigma}^{\mu\nu} \equiv \partial^{\{\mu}u^{\nu\}}$, $\tilde{\sigma}_l^{\mu\nu} \equiv \partial^{\{\mu}l^{\nu\}}$, $\tilde{\omega}_l^{\mu\nu} \equiv \frac{1}{2}\Xi^{\mu\alpha}\Xi^{\nu\beta}(\partial_\alpha l_\beta - \partial_\beta l_\alpha)$

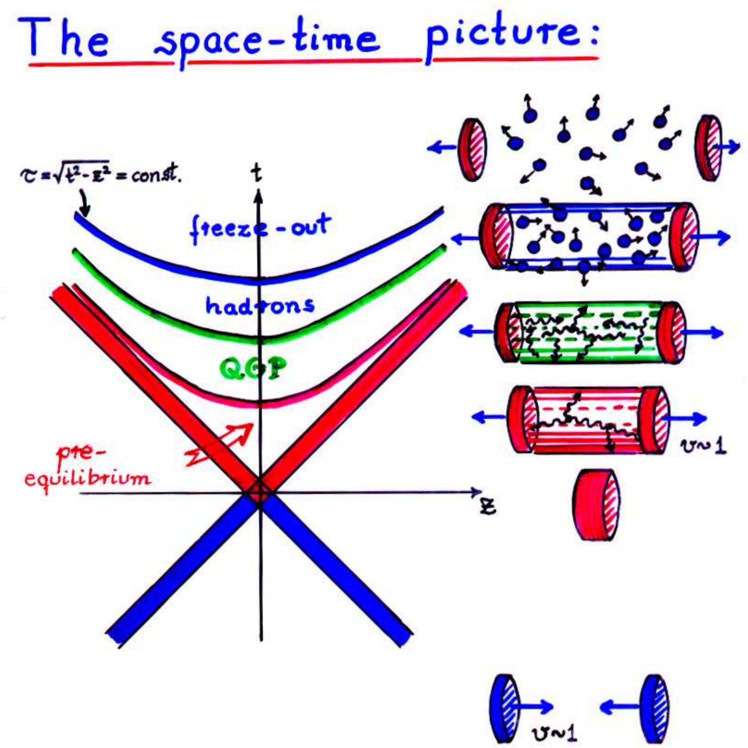
+ 9 relaxation equations for Π, n_l, $\hat{P}_l$, $V_\perp^\mu$, $W_{\perp l}^\mu$, $\tilde{\pi}^{\mu\nu}$

for details, see E. Molnár, H. Niemi, DHR, PRD 93 (2016) 11, 114025

Application to heavy-ion collisions (I)

Bjorken flow:

J.D. Bjorken, PRD 27 (1983) 140



"Pure" anisotropic fluid dynamics

$(\delta \hat{f}_{\mathrm{k}} \equiv 0 \iff \text{all } \rho_{rs}^{\mu_1 \cdots \mu_\ell} \equiv 0)$

$\Longrightarrow$ eqs. of motion for irreducible moments become eqs. of motion for moments $\hat{I}_{nrq}$:

$$\boxed{\partial_\tau \hat{I}_{i+j,j,0} + \frac{(j+1)\hat{I}_{i+j,j,0} + (i-1)\hat{I}_{i+j,j+2,0}}{\tau} = \hat{\mathcal{C}}_{i-1,j}}$$

$\Longrightarrow$ conservation equations:

$$i = 1, j = 0: \quad \partial_\tau \hat{n} + \frac{\hat{n}}{\tau} = 0$$

$$i = 2, j = 0: \quad \partial_\tau \hat{\epsilon} + \frac{\hat{\epsilon} + \hat{P}_l}{\tau} = 0$$

$\Longrightarrow$ 2 eqs., 3 unknowns: $\hat{\alpha}$, $\hat{\beta}_u$, ξ

$\Longrightarrow$ need add. eq. to close eqs. of motion!

$\Longrightarrow$ in principle, eq. of motion for any moment $\hat{I}_{i+j,j,0}$ suffices

$\Longrightarrow$ but which one is the best choice?

E. Molnár, H. Niemi, DHR, arXiv:1606.09019 [nucl-th]

Application to heavy-ion collisions (II)

assume relaxation-time approximation for collision term: $\hat{\mathcal{C}}_{i-1,j} \equiv -\dfrac{\hat{I}_{i+j,j,0} - I_{i+j,j,0}}{\tau_{\rm eq}}$

where $I_{i+j,j,0} = \lim_{\xi \to 0} \hat{I}_{i+j,j,0}$

$\Longrightarrow$ **study the following choices:**

(1) $i = 0,\ j = 2: \quad \partial_\tau \hat{P}_l + \dfrac{3\hat{P}_l - \hat{I}_{240}}{\tau} = -\dfrac{\hat{P}_l - I_{220}}{\tau_{\rm eq}}$

(2) $i = 3,\ j = 0: \quad \partial_\tau \hat{I}_{300} + \dfrac{\hat{I}_{300} - 2\hat{I}_{320}}{\tau} = -\dfrac{\hat{I}_{300} - I_{300}}{\tau_{\rm eq}}$

(3) $i = 1,\ j = 2: \quad \partial_\tau \hat{I}_{320} + \dfrac{3\hat{I}_{320}}{\tau} = -\dfrac{\hat{I}_{320} - I_{320}}{\tau_{\rm eq}}$

(4) $i = 0,\ j = 0: \quad \partial_\tau \hat{I}_{000} + \dfrac{\hat{I}_{000} - \hat{I}_{020}}{\tau} = -\dfrac{\hat{I}_{000} - I_{000}}{\tau_{\rm eq}}$

(5) $i = 0,\ j = 4: \quad \partial_\tau \hat{I}_{440} + \dfrac{5\hat{I}_{440} - \hat{I}_{460}}{\tau} = -\dfrac{\hat{I}_{440} - I_{440}}{\tau_{\rm eq}}$

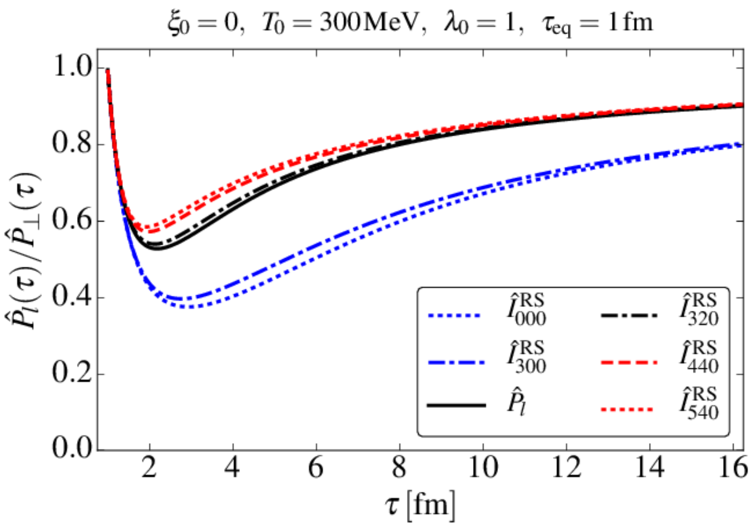
(6) $i = 1,\ j = 4: \quad \partial_\tau \hat{I}_{540} + \dfrac{5\hat{I}_{540}}{\tau} = -\dfrac{\hat{I}_{540} - I_{540}}{\tau_{\rm eq}}$

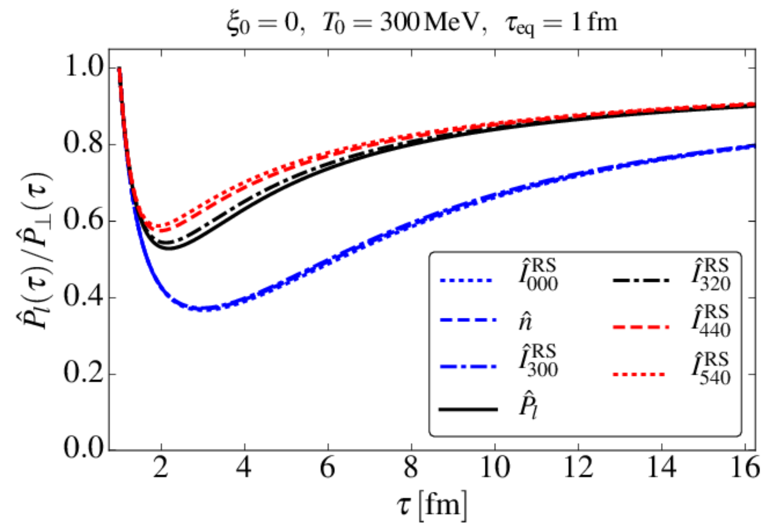
(7) **in case particle no. is not conserved:** $i = 1,\ j = 0: \quad \partial_\tau \hat{n} + \dfrac{\hat{n}}{\tau} = -\dfrac{\hat{n} - I_{100}}{\tau_{\rm eq}}$

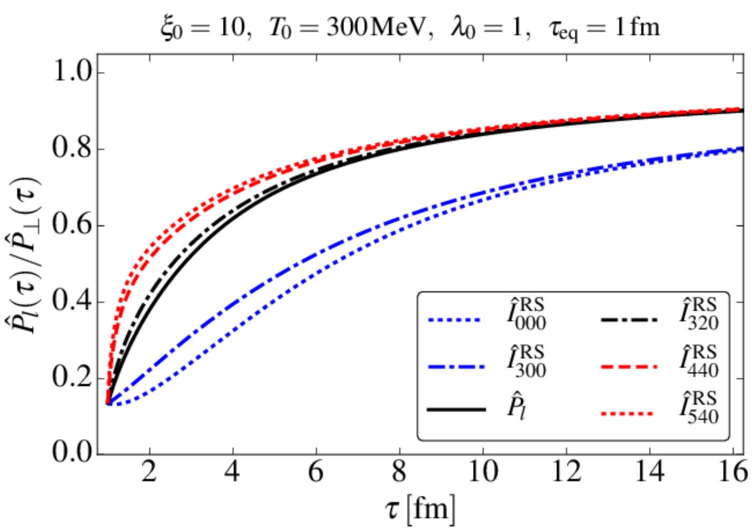
Note: different moments probe $\hat{f}_{0\rm k}$ in different regions of momentum space!

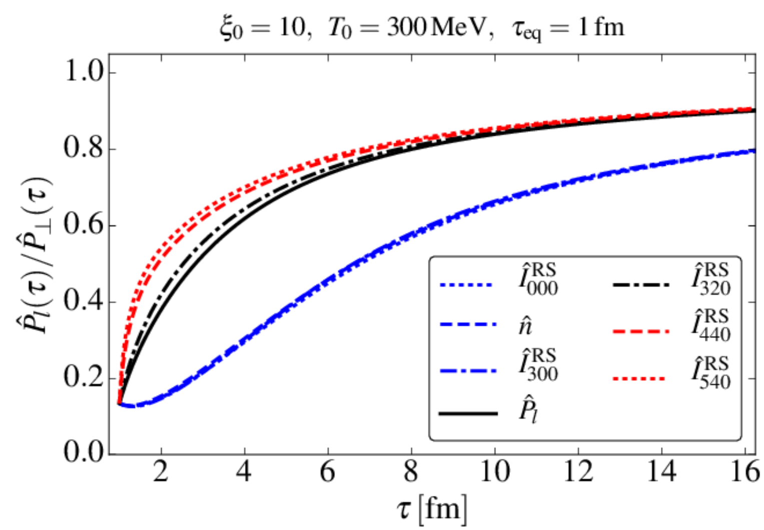
Application to heavy-ion collisions (III)

particle no. conservation: **no particle no. conservation:**







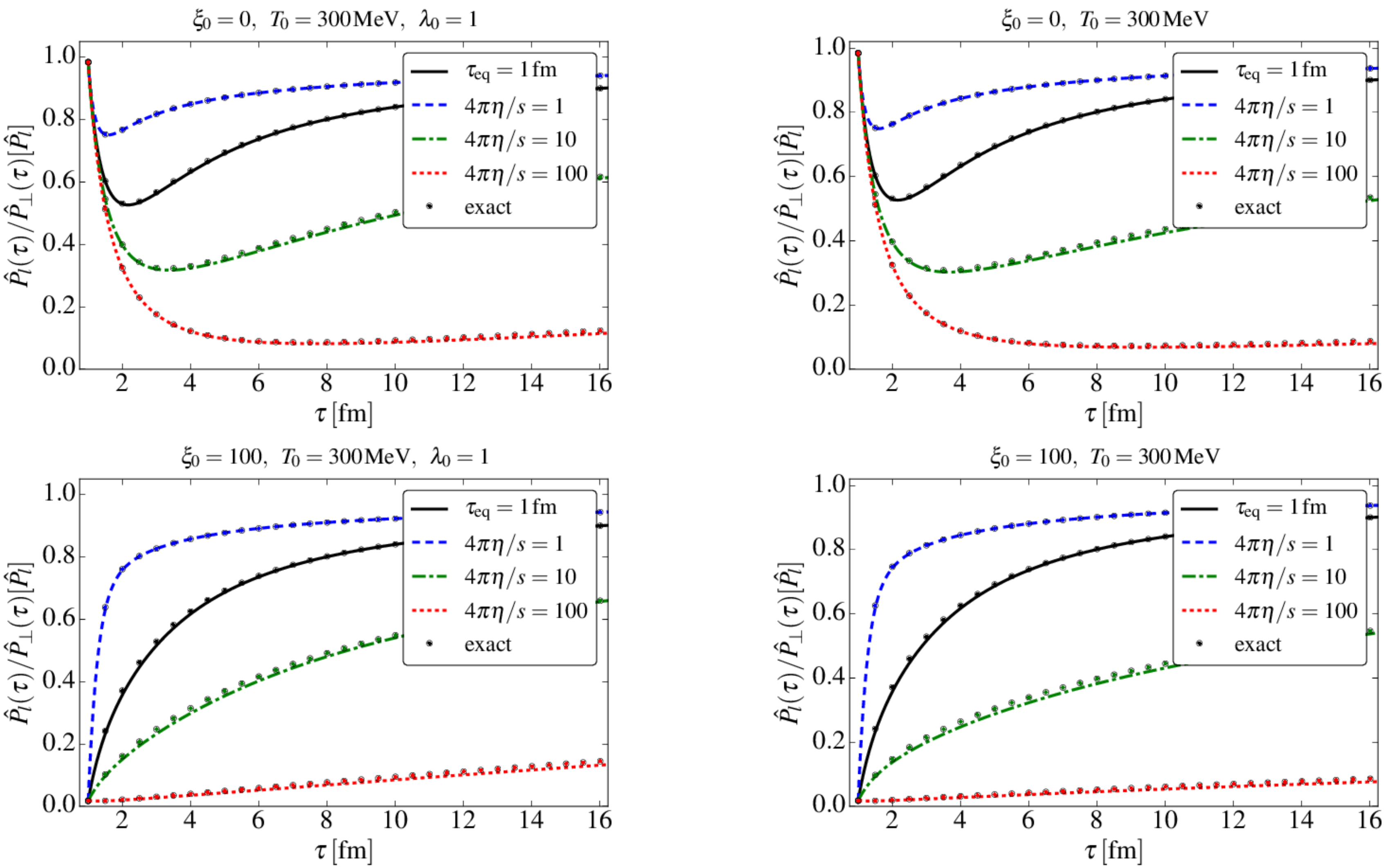


$\Longrightarrow$ **all cases (1) – (7) give different results!** $\Longrightarrow$ **which one is the best?**

Application to heavy-ion collisions (IV)

⟹ comparison of case (1) to solution of Boltzmann equation

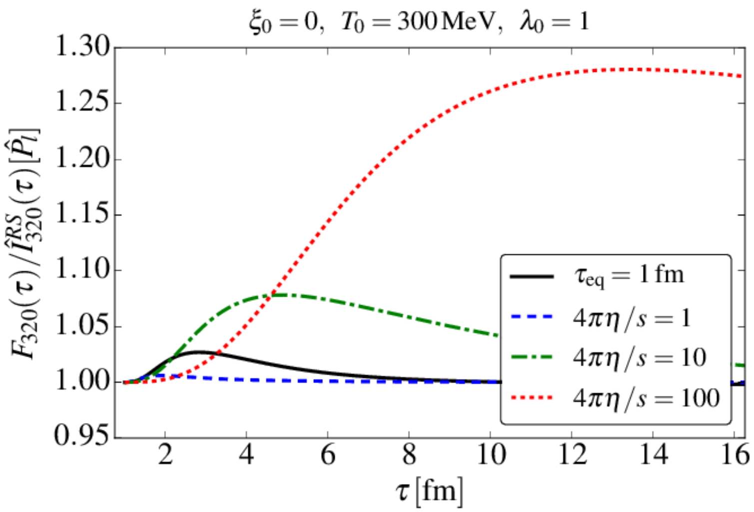
W. Florkowski, R. Ryblewski, M. Strickland, PRC 88 (2013) 024903

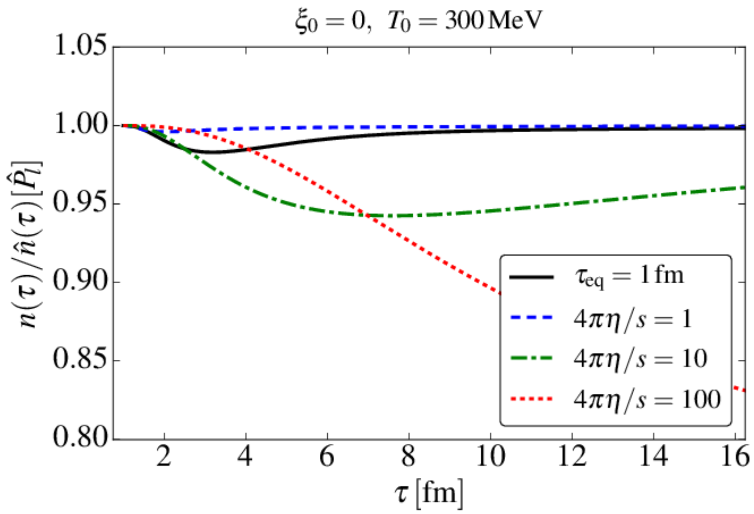


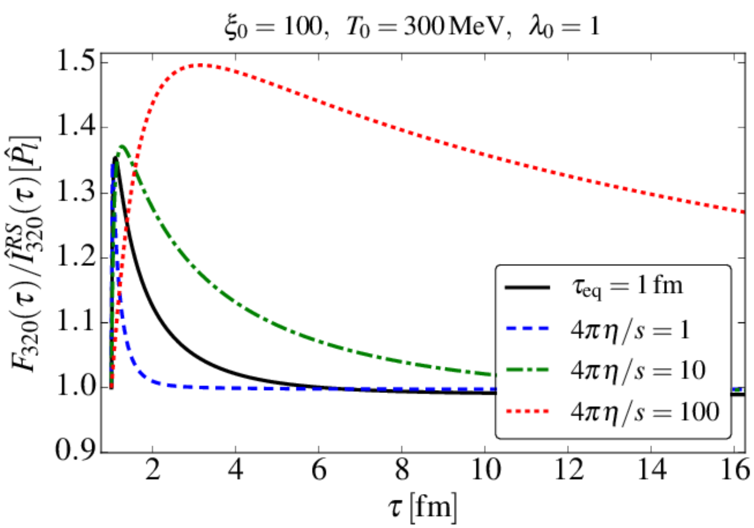
Application to heavy-ion collisions (V)

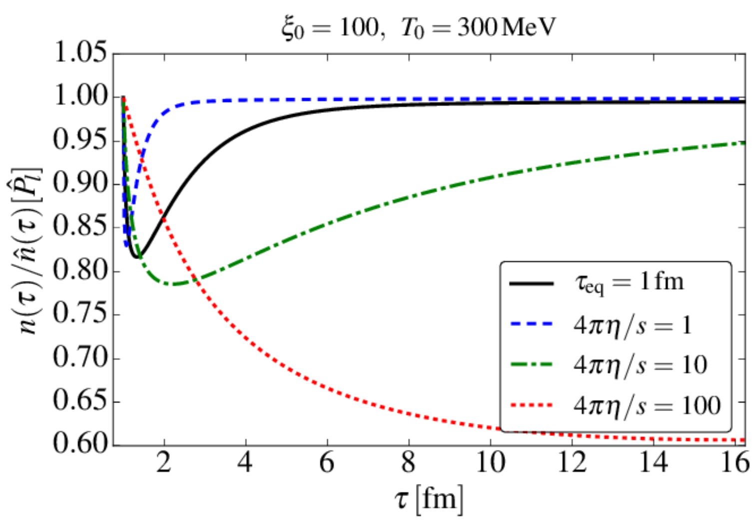
$\Longrightarrow$ **relaxation eq. for $\hat{P}_l$ gives best match to solution of Boltzmann equation!**

However: other moments not necessarily also agree well with Boltzmann eq.









Conclusions and Outlook

1. **Derivation of equations of motion of anisotropic dissipative fluid dynamics from Boltzmann equation**

   E. Molnár, H. Niemi, DHR, PRD 93 (2016) 11, 114025

   $\Longrightarrow$ **still need to do eigenmode analysis!**

2. **Closure of equations of motion of "pure" anisotropic fluid dynamics**

   $\Longrightarrow$ **best agreement to solution of Boltzmann equation provided by** $\hat{P}_l$

   but: not all moments agree with solution of Boltzmann equation

   E. Molnár, H. Niemi, DHR, arXiv:1606.09019 [nucl-th]

   $\Longrightarrow$ **need to improve** $\hat{f}_{0\mathrm{k}}$**?!**