Construction of Wave Functions of Highly Excited Multiquark States; N(1440), N(1520), N(1535) and Pentaquarks

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OUTLINE

- Introduction
- Construction of $q^4\bar{q}$ states
- Model results of q^3 excited state and $q^4\bar{q}$ ground state masses
- Summary

Roper Resonance $N_{1/2+}(1440)$

- In the traditional q^3 picture, the Roper $N_{1/2+}(1440)$ usually gets a mass ~ 100 MeV above the $N_{1/2-}(1535)$, but not 100 MeV below it.

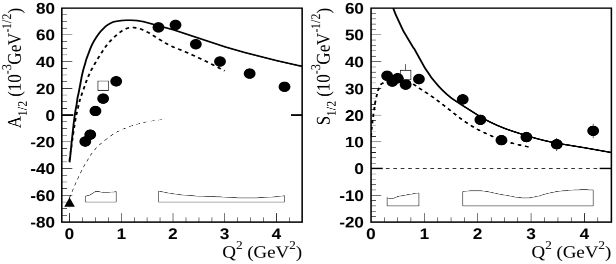
- Roper resonance is usually blamed sitting at a wrong place or intruding the q^3 spectrum.

- It has been studied in any possible picture: normal q^3 first radial excitation, $q^4\bar{q}$ pentaquark, q^3g hybrid, $q^3(q\bar{q})$ resonance...

- Still an open question.

Roper Resonance $N_{1/2+}(1440)$

Helicity amplitudes for the $\gamma^* p \to N(1440)$ transition. The thick curves correspond to quark models assuming that $N(1440)$ is a q^3 first radial excitation: dashed (Capstick and Keister, 1995), solid (Aznauryan, 2007). The thin dashed curves are obtained assuming that $N(1440)$ is a $q^3 g$ hybrid state (Li et al., 1992). Figure courtesy to Rev. Mod. Phys. **82**, 1095.



- The sign change in the helicity amplitude as a function of Q^2 suggests a node in the wave function and thus a radially excited state.

$N_{1/2-}(1535)$

- This resonance is observed at a mass expected in quark models
- Large couplings to the $N\eta$, $N\eta'$, $N\phi$ and $K\Lambda$ but small couplings to the $N\pi$ and $K\Sigma$ are claimed.
- A large $N\eta$ coupling invites speculation that it might be created dynamically as $N\eta - \Sigma K$ coupled channel effect.
- A large $N\phi$ coupling leads to the proposal that the $N_{1/2-}(1535)$ may have a large component of $uuds\bar{s}$ pentaquark states.

- That the pentaquark should be a color singlet demands that the color part of the pentaquark wave function must be a $[222]_1$ singlet.

$$\psi^c_{[222]}(q^4\overline{q}) = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline & \\ \hline \end{array} \tag{1}$$

- The color part of the antiquark in pentaquark states is a $[11]_3$ antitriplet

$$\psi^c_{[11]}(\overline{q}) = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \tag{2}$$

- The color wave function of the four-quark configuration must be a $[211]_3$ triplet

$$\psi^c_{[211]_\lambda}(q^4) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \psi^c_{[211]_\rho}(q^4) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \psi^c_{[211]_\eta}(q^4) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \tag{3}$$

q^4 Color Wave Functions

- q^4 color wave functions can be derived by applying the λ-, ρ- and η-type projection operators of the S_4 $\mathrm{IR}[211]$ in Yamanouchi basis,

$$\left|\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline R & R \\ \hline G \\ \cline{1-1} B \\ \cline{1-1}\end{array}\right\rangle = P_{[211]_\lambda}(RRGB) \Longrightarrow \psi^c_{[211]_\lambda}(R):$$

$$\frac{1}{\sqrt{16}}(2|RRGB\rangle - 2|RRBG\rangle - |GRRB\rangle - |RGRB\rangle - |BRGR\rangle$$
$$-|RBGR\rangle + |BRRG\rangle + |GRBR\rangle + |RBRG\rangle + |RGBR\rangle)$$

$$\left|\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline R & R \\ \hline G \\ \cline{1-1} B \\ \cline{1-1}\end{array}\right\rangle = P_{[211]_\rho}(RGRB) \Longrightarrow \psi^c_{[211]_\rho}(R):$$

$$\frac{1}{\sqrt{48}}(3|RGRB\rangle - 3|GRRB\rangle + 3|BRRG\rangle - 3|RBRG\rangle + 2|GBRR\rangle$$
$$-2|BGRR\rangle - |BRGR\rangle + |RBGR\rangle + |GRBR\rangle - |RGBR\rangle)$$

q^4 Color Wave Functions

$$\left|\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline R & R \\ \hline G \\ \cline{1-1} B \\ \cline{1-1}\end{array}\right\rangle = P_{[211]_\eta}(RGBR) \Longrightarrow \psi^c_{[211]_\eta}(R):$$

$$\frac{1}{\sqrt{6}}(|RGBR\rangle + |GBRR\rangle + |BRGR\rangle - |RBGR\rangle - |GRBR\rangle - |BGRR\rangle)$$

The singlet color wave function $\Psi^c_{[211]_j}$ $(j = \lambda, \rho, \eta)$ of pentaquarks is given by

$$\Psi^c_{[211]_j} = \frac{1}{\sqrt{3}}\left[\psi^c_{[211]_j}(R)\,\bar{R} + \psi^c_{[211]_j}(G)\,\bar{G} + \psi^c_{[211]_j}(B)\,\bar{B}\right]. \quad (4)$$

q^4 Wave Functions

- The total wave function of q^4 systems may be written in the general form,

$$\psi = \sum_{i,j=\lambda,\rho,\eta} a_{ij}\ \psi^{c}_{[211]_i}\psi^{osf}_{[31]_j} \tag{5}$$

with

$$\psi^{osf}_{[31]} = \sum_{i,j=S,A,\lambda,\rho,\eta} b_{ij}\ \psi^{o}_{[X]_i}\psi^{sf}_{[Y]_j}$$

$$\psi^{sf}_{[Z]} = \sum_{i,j=S,A,\lambda,\rho,\eta} c_{ij}\ \phi^{f}_{[X]_i}\chi^{s}_{[Y]_j}$$

- Possible configurations and the coefficients can be determined by applying the Yamanouchi-basis representations of the S_4 to the general forms.

q^4 Spatial-Flavor-Spin States

- Total wave function of q^4 systems

$$\psi = \frac{1}{\sqrt{3}}\left(\psi^{c}_{[211]_\lambda}\psi^{osf}_{[31]_\rho} - \psi^{c}_{[211]_\rho}\psi^{osf}_{[31]_\lambda} + \psi^{c}_{[211]_\eta}\psi^{osf}_{[31]_\eta}\right) \quad (6)$$

- Spatial-spin-flavor configurations:

$[31]_{OSF}$	
$[4]_O$	$[31]_{SF}$
$[1111]_O$	$[211]_{SF}$
$[22]_O$	$[31]_{SF}, [211]_{SF}$
$[211]_O$	$[31]_{SF}, [211]_{SF}, [22]_{SF}$
$[31]_O$	$[4]_{SF}, [31]_{SF}, [211]_{SF}, [22]_{SF}$

q^4 Spin-Flavor Configurations

	$[4]_{FS}$		
$[4]_{FS}[22]_F[22]_S$	$[4]_{FS}[31]_F[31]_S$	$[4]_{FS}[4]_F[4]_S$	
	$[31]_{FS}$		
$[31]_{FS}[31]_F[22]_S$	$[31]_{FS}[31]_F[31]_S$	$[31]_{FS}[31]_F[4]_S$	$[31]_{FS}[211]_F[22]_S$
$[31]_{FS}[211]_F[31]_S$	$[31]_{FS}[22]_F[31]_S$	$[31]_{FS}[4]_F[31]_S$	
	$[22]_{FS}$		
$[22]_{FS}[22]_F[22]_S$	$[22]_{FS}[22]_F[4]_S$	$[22]_{FS}[4]_F[22]_S$	$[22]_{FS}[211]_F[31]_S$
$[22]_{FS}[31]_F[31]_S$			
	$[211]_{FS}$		
$[211]_{FS}[211]_F[22]_S$	$[211]_{FS}[211]_F[31]_S$	$[211]_{FS}[211]_F[4]_S$	$[211]_{FS}[22]_F[31]_S$
$[211]_{FS}[31]_F[22]_S$	$[211]_{FS}[31]_F[31]_S$		

$q^3 Q\bar{Q}$ configuration

- q^3 and $Q\bar{Q}$ in color singlet state:

$$\begin{aligned}\Psi_{Octet}(q^3) &= \frac{1}{\sqrt{2}}\psi^c_{[111]}\psi^o_{[3]}(\phi_{[21]_\lambda}\chi_{[21]_\lambda} + \phi_{[21]_\rho}\chi_{[21]_\rho}),\\ \Psi_{Decuplet}(q^3) &= \psi^c_{[111]}\psi^o_{[3]}\phi_{[3]}\chi_{[3]}\end{aligned} \tag{7}$$

- Hidden color states, q^3 and $Q\bar{Q}$ in color [21] states:

$$\begin{aligned}\Psi(q^3) &= \frac{1}{\sqrt{2}}(\psi^c_{[21]_\lambda}\psi^{sf}_{[21]_\rho} - \psi^c_{[21]_\rho}\psi^{sf}_{[21]_\lambda}),\\ \psi^{sf}(q^3) &= \sum_{i,j} a_{ij}\psi^s_{[X]_i}\psi^f_{[Y]_j}\end{aligned} \tag{8}$$

with

$$\psi^s_{[X]_i} = \{\psi^s_{[3]}, \psi^s_{[21]_{\lambda,\rho}}\}, \quad \psi^f_{[Y]_j} = \{\psi^f_{[3]}, \psi^f_{[21]_{\lambda,\rho}}\} \tag{9}$$

and color wave function,

$$\Psi^c_{[222]}(q^3 Q\bar{Q}) = \frac{1}{\sqrt{8}}\sum_i \psi^c_{[21]_i}(q^3)\psi^c_{[21]_i}(Q\bar{Q}) \tag{10}$$

Spatial Wave Functions

- Harmonic oscillator wave functions may be considered as the first order approximation for pentaquark systems.
- A complete basis of certain permutation symmetry may be constructed from harmonic oscillator wave functions of the most simple H,

$$H = \frac{p_\lambda^2}{2m} + \frac{p_\rho^2}{2m} + \frac{p_\eta^2}{2m} + \frac{p_\xi^2}{2m} + \frac{1}{2}C\left(\lambda^2 + \rho^2 + \eta^2 + \xi^2\right) \qquad (11)$$

where

$$\begin{aligned}
\vec{\rho} &= \frac{1}{\sqrt{2}}(\vec{r_1} - \vec{r_2}) \\
\vec{\lambda} &= \frac{1}{\sqrt{6}}(\vec{r_1} + \vec{r_2} - 2\vec{r_3}) \\
\vec{\eta} &= \frac{1}{\sqrt{12}}(\vec{r_1} + \vec{r_2} + \vec{r_3} - 3\vec{r_4}) \\
\vec{\xi} &= \frac{1}{\sqrt{20}}(\vec{r_1} + \vec{r_2} + \vec{r_3} + \vec{r_4} - 4\vec{r_5})
\end{aligned} \qquad (12)$$

Spatial Wave Functions

Spatial wave functions take the general form,

$$\begin{aligned}\Psi^{o}_{NLM} = & \sum_{n_\lambda,n_\rho,n_\eta,n_\xi,l_\lambda,l_\rho,l_\eta,l_\xi} A(n_\lambda,n_\rho,n_\eta,n_\xi,l_\lambda,l_\rho,l_\eta,l_\xi) \\ & \cdot\Psi_{n_\lambda l_\lambda m_\lambda}(\vec{\lambda})\Psi_{n_\rho l_\rho m_\rho}(\vec{\rho})\Psi_{n_\eta l_\eta m_\eta}(\vec{\eta})\Psi_{n_\xi l_\xi m_\xi}(\vec{\xi}) \\ & \cdot C(l_\lambda,l_\rho,m_\lambda,m_\rho,l_{\lambda\rho},m_{\lambda\rho}) \\ & \cdot C(l_{\lambda\rho},l_\eta,m_{\lambda\rho},m_\eta,l_{\lambda\rho\eta},m_{\lambda\rho\eta}) \\ & \cdot C(l_{\lambda\rho\eta},l_\xi,m_{\lambda\rho\eta},m_\xi,LM) \end{aligned} \quad (13)$$

with $N = 2(n_\lambda + n_\rho + n_\eta + n_\xi) + l_\lambda + l_\rho + l_\eta + l_\xi$

- The coefficients A are determined by applying the Yamanouchi basis representations of the S_4. Spatial wave functions with [4], [31], [22], [211] and [1111] symmetries can be derived easily.

- Hamiltonian for a N-quark system:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i^0} + C \sum_{i<j}^{N} (\vec{r}_i - \vec{r}_j)^2 + \sum_{i=1}^{N} m_i^0 + H_{hyp} \tag{14}$$

$$H_{hyp}^{OGE} = -C_G \sum_{i<j} \frac{\lambda_i^C \cdot \lambda_j^C}{m_i m_j} \vec{\sigma}_i \cdot \vec{\sigma}_j \tag{15}$$

- m_i^0 stands for "bare" quark masses, m_i are dressed quark masses resulted from m_i^0 and the ground-state energy of harmonic oscillation.
- Model parameters determined by fitting theoretical results to 4 vector meson isospin states, 8 baryon isospin states, $J/\psi(1S)$ and $\Upsilon(1S)$:
  $m_u^0(m_u) = 53(362)$ MeV, $m_s^0(m_s) = 361(532)$ MeV
  $m_c^0(m_c) = 1480(1568)$ MeV, $m_b^0(m_b) = 4689(4739)$ MeV
  $w_0 = \sqrt{\frac{2C}{m_u}} = 138$ MeV, $C_m = C_G/m_u^2 = 19.0$ MeV

Mass of excited non-strange q^3 states

States $\Psi(N,L)$	J^P	Mass MeV
$\Psi_{\text{Singlet}}(1,1)$	$\frac{1}{2}^-, \frac{3}{2}^-$	1174
$\Psi^{(1)}_{\text{Octet}}(1,1)$	$\frac{1}{2}^-, \frac{3}{2}^-$	1174
$\Psi^{(2)}_{\text{Octet}}(1,1)$	$\frac{1}{2}^-, \frac{3}{2}^-, \frac{5}{2}^-$	1477
$\Psi_{\text{Decuplet}}(1,1)$	$\frac{1}{2}^-, \frac{3}{2}^-$	1174
$\Psi_{\text{Singlet}}(2,0)$	$\frac{1}{2}^+$	1413
$\Psi^{(1)}_{\text{Octet}}(2,0)$	$\frac{1}{2}^+$	1413
$\Psi^{(2)}_{\text{Octet}}(2,0)$	$\frac{3}{2}^+$	1717
$\Psi^{(3)}_{\text{Octet}}(2,0)$	$\frac{1}{2}^+$	1413
$\Psi^{(1)}_{\text{Decuplet}}(2,0)$	$\frac{1}{2}^+$	1413
$\Psi^{(2)}_{\text{Decuplet}}(2,0)$	$\frac{3}{2}^+$	1717

Ground state pentaquark $q^4\bar{q}$ masses

$q^4\bar{q}$ Configurations	Spin (or J)	$M(q^4\bar{q})$ (MeV)
$\Psi^{sf}_{[31]_{FS}[4]_F[31]_S}(q^4\overline{q})$	$\frac{1}{2}$, $\frac{3}{2}$	3024, 2720
$\Psi^{sf}_{[31]_{FS}[31]_F[4]_S}(q^4\overline{q})$	$\frac{3}{2}$, $\frac{5}{2}$	2467, 2720
$\Psi^{sf}_{[31]_{FS}[31]_F[31]_S}(q^4\overline{q})$	$\frac{1}{2}$, $\frac{3}{2}$	2568, 2493
$\Psi^{sf}_{[31]_{FS}[31]_F[22]_S}(q^4\overline{q})$	$\frac{1}{2}$	2467
$\Psi^{sf}_{[31]_{FS}[22]_F[31]_S}(q^4\overline{q})$	$\frac{1}{2}$, $\frac{3}{2}$	2113, 2493

Ground state pentaquark $q^3 s\bar{s}$ masses

$q^4\bar{q}$ Configurations	Spin (or J)	$M(q^4\bar{q})$ (MeV)
$\Psi^{sf}_{[31]_{FS}[4]_F[31]_S}(q^3 s\bar{s})$	$\frac{1}{2}, \frac{3}{2}$	3165, 2949
$\Psi^{sf}_{[31]_{FS}[31]_F[4]_S}(q^3 s\bar{s})$	$\frac{3}{2}, \frac{5}{2}$	2755, 2917
$\Psi^{sf}_{[31]_{FS}[31]_F[31]_S}(q^3 s\bar{s})$	$\frac{1}{2}, \frac{3}{2}$	2801, 2756
$\Psi^{sf}_{[31]_{FS}[31]_F[22]_S}(q^3 s\bar{s})$	$\frac{1}{2}$	2733
$\Psi^{sf}_{[31]_{FS}[211]_F[31]_S}(q^3 s\bar{s})$	$\frac{1}{2}, \frac{3}{2}$	2322, 2585
$\Psi^{sf}_{[31]_{FS}[211]_F[22]_S}(q^3 s\bar{s})$	$\frac{1}{2}$	2491
$\Psi^{sf}_{[31]_{FS}[22]_F[31]_S}(q^3 s\bar{s})$	$\frac{1}{2}, \frac{3}{2}$	2441, 2711

Ground state pentaquarks $q^3 c\bar{c}$ in $q^3 Q\bar{Q}$ configuration

$q^3 Q\bar{Q}$ Configurations	J^P	$M(q^3 c\bar{c})$(MeV)
$\Psi^{csf}_{[111]_C [21]_F [21]_S}(q^3 c\bar{c})$	$\frac{1}{2}^-, \frac{3}{2}^-$	4280, 4301
$\Psi^{csf}_{[111]_C [21]_F [3]_S}(q^3 c\bar{c})$	$\frac{3}{2}^-, \frac{5}{2}^-$	4493, 4606
$\Psi^{csf}_{[21]_C [21]_F [21]_S}(q^3 c\bar{c})$	$\frac{1}{2}^-, \frac{3}{2}^-$	4412, 4415
$\Psi^{csf}_{[21]_C [3]_F [21]_S}(q^3 c\bar{c})$	$\frac{1}{2}^-, \frac{3}{2}^-$	4640, 4638
$\Psi^{csf}_{[21]_C [21]_F [3]_S}(q^3 c\bar{c})$	$\frac{3}{2}^-, \frac{5}{2}^-$	4488, 4503

Ground state pentaquarks $q^3b\bar{b}$ in $q^3Q\bar{Q}$ configuration

$q^3Q\bar{Q}$ Configurations	J^P	$M(q^3b\bar{b})$(MeV)
$\Psi^{csf}_{[111]_C[21]_F[21]_S}(q^3b\bar{b})$	$\frac{1}{2}^-, \frac{3}{2}^-$	10664, 10667
$\Psi^{csf}_{[111]_C[21]_F[3]_S}(q^3b\bar{b})$	$\frac{3}{2}^-, \frac{5}{2}^-$	10868,10971
$\Psi^{csf}_{[21]_C[21]_F[21]_S}(q^3b\bar{b})$	$\frac{1}{2}^-, \frac{3}{2}^-$	10780, 10782
$\Psi^{csf}_{[21]_C[3]_F[21]_S}(q^3b\bar{b})$	$\frac{1}{2}^-, \frac{3}{2}^-$	11008,11008
$\Psi^{csf}_{[21]_C[21]_F[3]_S}(q^3b\bar{b})$	$\frac{3}{2}^-, \frac{5}{2}^-$	10856, 10862

Discussion

- The work gives a mass about 1410 MeV for non-strange q^3 first radial excited states. It may imply that the Roper resonance is mainly a q^3 state.

- Assuming that $N(1535)$ and $N(1520)$ have a large $q^3 s\bar{s}$ and $q^4\bar{q}$ component of the spin-flavor configuration $[22]_F[31]_S$, then we have from $M(q^3) = 1174$ MeV, $M(q^3 s\bar{s})_{s=1/2} = 2441$ MeV and $M(q^4\bar{q})_{s=3/2} = 2493$ MeV,

States	J^P	$q^3\%$	$q^4\bar{q}\%$
$N(1535)$	$\frac{1}{2}^-$	71.5	28.5
$N(1520)$	$\frac{3}{2}^-$	73.8	26.2

- The work gives masses of $4280 \sim 4640$ MeV for ground state pentaquarks $q^3 c\bar{c}$. It is consistent with the LHCb observation of $P_c^+(4380)$ and $P_c^+(4450)$ (Indeed, Kai did calculations half a year before we saw LHCb report).

- The work predicts that ground state $q^3 b\bar{b}$ may have masses around 11 GeV.

Thank you for your attention!

Pentaquark Spatial Wave Function, $NLM = 322$

- Symmetric:

$$\Psi^S = \frac{1}{3}\,[-\Psi_{021}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{011}(\xi) + \sqrt{2}\Psi_{022}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{010}(\xi)$$
$$-\Psi_{000}(\lambda)\Psi_{021}(\rho)\Psi_{000}(\eta)\Psi_{011}(\xi) + \sqrt{2}\Psi_{000}(\lambda)\Psi_{022}(\rho)\Psi_{000}(\eta)\Psi_{010}(\xi)$$
$$-\Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{021}(\eta)\Psi_{011}(\xi) + \sqrt{2}\Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{022}(\eta)\Psi_{010}(\xi)\,]$$

- Antisymmetric: Non
- λ and ρ types of [22]: Non
- λ, ρ and η types of [211]: Non

Pentaquark Spatial Wave Function, $NLM = 322$

First Set of λ, ρ and η types of [31]:

$$\Psi^{\lambda[31]} = \frac{1}{\sqrt{6}}[\sqrt{2}\Psi_{010}(\lambda)\Psi_{022}(\rho)\Psi_{000}(\eta)\Psi_{000}(\xi) - \Psi_{011}(\lambda)\Psi_{021}(\rho)\Psi_{000}(\eta)\Psi_{000}(\xi)$$
$$+\sqrt{2}\Psi_{010}(\lambda)\Psi_{000}(\rho)\Psi_{022}(\eta)\Psi_{000}(\xi) - \Psi_{011}(\lambda)\Psi_{000}(\rho)\Psi_{021}(\eta)\Psi_{000}(\xi)]$$

$$\Psi^{\rho[31]} = \frac{1}{\sqrt{6}}[\sqrt{2}\Psi_{022}(\lambda)\Psi_{010}(\rho)\Psi_{000}(\eta)\Psi_{000}(\xi) - \Psi_{021}(\lambda)\Psi_{011}(\rho)\Psi_{000}(\eta)\Psi_{000}(\xi)$$
$$+\sqrt{2}\Psi_{000}(\lambda)\Psi_{010}(\rho)\Psi_{022}(\eta)\Psi_{000}(\xi) - \Psi_{000}(\lambda)\Psi_{011}(\rho)\Psi_{021}(\eta)\Psi_{000}(\xi)]$$

$$\Psi^{\eta[31]} = \frac{1}{\sqrt{6}}[\sqrt{2}\Psi_{022}(\lambda)\Psi_{000}(\rho)\Psi_{010}(\eta)\Psi_{000}(\xi) - \Psi_{021}(\lambda)\Psi_{000}(\rho)\Psi_{011}(\eta)\Psi_{000}(\xi)$$
$$+\sqrt{2}\Psi_{000}(\lambda)\Psi_{022}(\rho)\Psi_{010}(\eta)\Psi_{000}(\Xi) - \Psi_{000}(\lambda)\Psi_{021}(\rho)\Psi_{011}(\eta)\Psi_{000}(\Xi)]$$

Pentaquark Spatial Wave Function, $NLM = 322$

Second Set of λ, ρ and η types of [31]:

$$\Psi^{\lambda[31]} = \frac{1}{\sqrt{3}}[\sqrt{2}\Psi_{010}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{022}(\xi) - \Psi_{011}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{021}(\xi)]$$

$$\Psi^{\rho[31]} = \frac{1}{\sqrt{3}}[\sqrt{2}\Psi_{000}(\lambda)\Psi_{010}(\rho)\Psi_{000}(\eta)\Psi_{022}(\xi) - \Psi_{000}(\lambda)\Psi_{011}(\rho)\Psi_{000}(\eta)\Psi_{021}(\xi)]$$

$$\Psi^{\eta[31]} = \frac{1}{\sqrt{3}}[\sqrt{2}\Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{010}(\eta)\Psi_{022}(\xi) - \Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{011}(\eta)\Psi_{021}(\xi)]$$

Pentaquark Spatial Wave Function, Symmetric $NLM = 444$

$$\Psi_1^S = \Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{044}(\xi)$$

$$\Psi_2^S = \frac{1}{\sqrt{3}}[\Psi_{022}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{022}(\xi) + \Psi_{000}(\lambda)\Psi_{022}(\rho)\Psi_{000}(\eta)\Psi_{022}(\xi) + \Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{022}(\eta)\Psi_{022}(\xi)]$$

$$\Psi_3^S = \sqrt{\frac{1}{17}}[\Psi_{033}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{011}(\xi) - \sqrt{7}\Psi_{011}(\lambda)\Psi_{022}(\rho)\Psi_{000}(\eta)\Psi_{011}(\xi) + \sqrt{2}\Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{033}(\eta)\Psi_{011}(\xi) - \sqrt{\frac{7}{2}}\Psi_{022}(\lambda)\Psi_{000}(\rho)\Psi_{011}(\eta)\Psi_{011}(\xi) - \sqrt{\frac{7}{2}}\Psi_{000}(\lambda)\Psi_{022}(\rho)\Psi_{011}(\eta)\Psi_{011}(\xi)]$$

$$\Psi_4^S = \sqrt{\frac{5}{57}}[\Psi_{044}(\lambda)\Psi_{000}(\rho)\Psi_{000}(\eta)\Psi_{000}(\xi) + \Psi_{000}(\lambda)\Psi_{044}(\rho)\Psi_{000}(\eta)\Psi_{000}(\xi) + \Psi_{000}(\lambda)\Psi_{000}(\rho)\Psi_{044}(\eta)\Psi_{000}(\xi) + \sqrt{\frac{14}{5}}\Psi_{022}(\lambda)\Psi_{022}(\rho)\Psi_{000}(\eta)\Psi_{000}(\xi) + \sqrt{\frac{14}{5}}\Psi_{022}(\lambda)\Psi_{000}(\rho)\Psi_{022}(\eta)\Psi_{000}(\xi) + \sqrt{\frac{14}{5}}\Psi_{000}(\lambda)\Psi_{022}(\rho)\Psi_{022}(\eta)\Psi_{000}(\xi)]$$

Matrix Representations of S_4

- S_4 [211]

$$D^{[211]}(12) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; D^{[211]}(23) = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$D^{[211]}(34) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/3 & 2\sqrt{2}/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{pmatrix}$$

- S_4 [31]

$$D^{[31]}(12) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; D^{[31]}(23) = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D^{[31]}(34) = \begin{pmatrix} 1/3 & 0 & 2\sqrt{2}/3 \\ 0 & 1 & 0 \\ 2\sqrt{2}/3 & 0 & -1/3 \end{pmatrix}$$

q^3 Baryon Excited States:

(1) $N, L = 1, 1$:

$$\begin{aligned}
\Psi^{(q^3)}_{Singlet} &= \frac{1}{\sqrt{2}}\psi^c_{[111]}\Phi_A(\phi^1_{1m\lambda}\chi_\rho - \phi^1_{1m\rho}\chi_\lambda), \\
\Psi^{(q^3)}_{Octet1} &= \frac{1}{2}\psi^c_{[111]}[\phi^1_{1m\rho}(\Phi_\lambda\chi_\rho + \Phi_\rho\chi_\lambda) \\
&\quad +\phi^1_{1m\lambda}(\Phi_\rho\chi_\rho - \Phi_\lambda\chi_\lambda)], \\
\Psi^{(q^3)}_{Octet2} &= \frac{1}{\sqrt{2}}\psi^c_{[111]}\chi_S(\phi^1_{1m\lambda}\Phi_\lambda + \phi^1_{1m\rho}\Phi_\rho), \\
\Psi^{(q^3)}_{Decuplet} &= \frac{1}{\sqrt{2}}\psi^c_{[111]}\Phi_S(\phi^1_{1m\lambda}\chi_\lambda + \phi^1_{1m\rho}\chi_\rho)
\end{aligned} \tag{17}$$

(2) $N, L = 2, 0$ (spatial part symmetric):

$$\begin{aligned} \Psi^{(q^3)}_{Octet} &= \frac{1}{\sqrt{2}}\psi^c_{[111]}\phi^2_{00S}(\Phi_\rho\chi_\rho + \Phi_\lambda\chi_\lambda), \\ \Psi^{(q^3)}_{Decuplet} &= \psi^c_{[111]}\phi^2_{00S}\Phi_S\chi_S \end{aligned} \tag{18}$$

(3) $N, L = 2, 0$ (spatial part mixed symmetric):

$$\begin{aligned} \Psi^{(q^3)}_{Singlet} &= \frac{1}{\sqrt{2}}\psi^c_{[111]}\Phi_A(\phi^2_{00\lambda}\chi_\rho - \phi^2_{00\rho}\chi_\lambda), \\ \Psi^{(q^3)}_{Octet1} &= \frac{1}{2}\psi^c_{[111]}[\phi^2_{00\rho}(\Phi_\lambda\chi_\rho + \Phi_\rho\chi_\lambda) \\ &\quad +\phi^2_{00\lambda}(\Phi_\rho\chi_\rho - \Phi_\lambda\chi_\lambda)], \\ \Psi^{(q^3)}_{Octet2} &= \frac{1}{\sqrt{2}}\psi^c_{[111]}\chi_S(\phi^2_{00\lambda}\Phi_\lambda + \phi^2_{00\rho}\Phi_\rho), \\ \Psi^{(q^3)}_{Decuplet} &= \frac{1}{\sqrt{2}}\psi^c_{[111]}\Phi_S(\phi^2_{00\lambda}\chi_\lambda + \phi^2_{00\rho}\chi_\rho) \end{aligned} \tag{19}$$