The QCD equation of state at weak coupling

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Probing the Extremes of Matter with Heavy Ions
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Table of contents

1 Setup
   • Bulk thermodynamics of QCD
   • Perturbative input

2 Small $\theta$: Hot quark gluon plasma
   • Dimensional reduction
   • Perturbative results for the EoS
   • Finite density effects

3 $\theta \approx \pi/2$: Cold quark matter
   • Introduction: Nuclear matter EoS
   • Weak coupling techniques at high density
   • Towards the saturation density

4 Conclusions
Table of contents

1 Setup
   • Bulk thermodynamics of QCD
   • Perturbative input

2 Small $\theta$: Hot quark gluon plasma
   • Dimensional reduction
   • Perturbative results for the EoS
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4 Conclusions
Equilibrium thermodynamics

Conceptually simple goal: Evaluate the grand potential of QCD

\[
\Omega(T, \{\mu_f\}, \{m_f\}) = -T \log \int D\bar{\psi} D\psi DA_\mu e^{-\int_0^\beta d\tau \int d^3x L_{\text{QCD}}}
\]

\[
L_{\text{QCD}} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \sum_f \bar{\psi}_f (\gamma_\mu D_\mu + m_f - \mu_f \gamma_0) \psi_f
\]

Bullk equilibrium thermodynamics from \(\Omega\):

\[
\begin{align*}
pV &= -\Omega \\
sV &= -\partial_T \Omega \\
n_fV &= -\partial_{\mu_f} \Omega \\
\varepsilon &= -p + Ts + \mu_f n_f \\
\langle \bar{\psi}_f \psi_f \rangle V &= \partial_{m_f} \Omega
\end{align*}
\]
Equilibrium thermodynamics

Unfortunately, QCD is a complicated theory, and no single method covers entire phase diagram ⇒ Need combination of (and interpolation between) several.
Corners of the phase diagram

Parametrize the phase diagram in terms of radial and angular variables: \( r \equiv \sqrt{T^2 + \frac{\mu_B^2}{9\pi^2}} \), \( \theta \equiv \arctan\frac{\mu_B}{T} \)

- \( r \) measures, how strongly coupled the system is
  - \( r \lesssim 100 \text{ MeV} \): Confinement; hadron resonance gas model, chiral effective theories, ...
  - 100 MeV \( \lesssim r \lesssim 500 \text{ MeV} \): Nonperturbative phase transition dynamics; lattice QCD, effective theories, models
  - \( r \gtrsim 500 \text{ MeV} \): Deconfinement, weakly interacting quasiparticles; weak coupling methods

- \( \theta \) separates two distinct physically interesting regimes
  - \( \theta \lesssim 1 \): Quark-gluon plasma; heavy ion collisions, early universe
  - \( \theta \approx \pi/2 \): Cold nuclear and quark matter; neutron stars
Weak coupling methods

In this talk: Try to extend perturbatively determined EoS to as low energy densities as possible

- $\theta \lesssim 1$: Complement lattice simulations at very high $T$ and/or $\mu \neq 0$
  - No sign problem: $\mu = 0$ results straightforwardly extendable to finite density
  - Guide for lattice simulations as $T \to \infty$

- $\theta \approx \pi/2$: Constrain nuclear matter EoSs above saturation density by providing information of the high $r$ region
  - Challenges: Technically complicated, in particular in the presence of quark pairing effects
Table of contents

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Energy scales in a high temperature plasma

At $T \gg T_c$, hierarchy of three length scales in the QGP:

- $\lambda \sim 1/(\pi T)$: Wavelength of thermal fluctuations, inverse effective mass of non-static field modes ($p_0 \neq 0$)
  - $n_b(E)g^2(T) \sim g^2(T) \Rightarrow$ Contributes to the EoS perturbatively (naive loop expansion)

- $\lambda \sim 1/(gT)$: Screening length of static color electric fluctuations, inverse thermal (‘Debye’) mass of $A_0$
  - $n_b(E)g^2(T) \sim g(T) \Rightarrow$ Physics somewhat perturbative at high $T$
  - Requires resummation of EoS at three loop order $\Rightarrow g^3, g^4 \ln g, ...$

- $\lambda \sim 1/(g^2 T)$: Nonperturbative screening length of static color magnetic fluctuations, inverse ‘magnetic mass’
  - $n_b(E)g^2(T) \sim g^0(T) \Rightarrow$ Physics non-perturbative at high $T$
  - Invalidates pert. expansion of EoS at four loops: ‘Linde’ problem

No further length scales due to confinement
Thermodynamics via dimensional reduction

- Scale hierarchy $\Rightarrow$ Natural to integrate out massive (non-static) modes (Appelquist, Pisarski)
  - Effective description accurate for $\Delta x \gtrsim 1/(gT)$

Result: 3d eff. thy for static dof’s (Kajantie et al; Braaten, Nieto):

\[
\mathcal{L}_{\text{EQCD}} = g_{\text{E}}^{-2}\left\{ \frac{1}{2} \text{Tr} F_{ij}^2 + \text{Tr}[(D_i A_0)^2] + m_{\text{E}}^2 \text{Tr} A_0^2 \right. \\
\left. + \lambda_{\text{E}} \text{Tr} A_0^4 \right\} + \delta \mathcal{L}_{\text{E}},
\]

\[
g_{\text{E}} \equiv \sqrt{T}g, \quad m_{\text{E}} \sim gT, \quad \lambda_{\text{E}} \sim g^2
\]
Thermodynamics via dimensional reduction

- EQCD valuable tool in reorganizing perturbation theory
  - No need for resummations in full theory, when dimensional reduction applicable
  - IR sector described by EQCD: Non-perturbative physics available from simulations in a 3d theory

\[
p_{\text{QCD}}(T, \mu) = p_E(T, \mu) + \frac{T}{V} \ln \int D A_i^a D A_0^a \exp \left\{ -S_E \right\}
\]

- Finite \( \mu \) has only minor effects on structure of effective theory as long as \( m_D \lesssim T \Leftrightarrow g \mu \lesssim T \) (Ipp, Kajantie, Rebhan, AV)
  - \( p_E \) and other 3d parameters now functions of \( T, \mu \)
  - One new operator generated in Lagrangian
The pressure at $\mu = 0$

$$p_{\text{QCD}} = T^4 \left\{ p_0(\mu/T) + g^2 p_2(\mu/T) + g^3 p_3(\mu/T) + g^4 \ln g \tilde{p}_4(\mu/T) \\
+ g^4 p_4(\mu/T) + g^5 p_5(\mu/T) + g^6 \ln g \tilde{p}_6(\mu/T) + g^6 p_6(\mu/T) + \cdots \right\}$$

First non-perturbative contributions from scale $g^2 T$, requiring 3d lattice simulations and a complicated conversion of results to continuum regularization (di Renzo et al.)

Contribution from scale $gT$ computed perturbatively in EQCD (Kajantie, Laine, Rummukainen, Schröder)

Contribution of scale $\pi T$ remaining, available through strict loop expansion of full theory pressure up to 4 loops; only $\mathcal{O}(N_f^3)$ term known (Gynther, Kurkela, AV)
The pressure at $\mu = 0$

\[
p_{QCD} = T^4 \left\{ p_0(\mu/T) + g^2 p_2(\mu/T) + g^3 p_3(\mu/T) + g^4 \ln g \tilde{p}_4(\mu/T) \right. \\
+ \left. g^4 p_4(\mu/T) + g^5 p_5(\mu/T) + g^6 \ln g \tilde{p}_6(\mu/T) + g^6 p_6(\mu/T) + \cdots \right\}
\]

Fitting unknown $O(g^6)$ term to lattice results and keeping EQCD parameters unexpanded gives an almost perfect match down to $T = 2T_c$ (Laine, Schröder)
Comparison with lattice

DR results in accordance with HTLpt (Andersen, Strickland, Su) and high-$T$ lattice data (Endrodi et al.) — $\mu = 0$ story seems fairly complete on the perturbative side.
Towards finite density: Susceptibilities

$O(g^6 \ln g)$ EoS generalized to finite density (AV): $\mu$-dependent part converges better than the $\mu = 0$ one due to absence of purely gluonic contributions

Natural observable allowing comparison with lattice data: Quark number susceptibilities

$$\chi_{ijk} \equiv - \left. \frac{\partial^n \Omega(T, \{\mu_f\}, \{m_f\})}{\partial \mu_u \partial \mu_d \partial \mu_s} \right|_{\mu_f=0}, \quad n = i + j + k$$

Allows comparison of various resummation schemes (DR, HTLpt,...)
Towards finite density: Susceptibilities

Recent comparison of DR and HTLpt results for linear quark number susceptibility $\chi_{uu}$ (Andersen, Mogliacci, Su, AV) with lattice results shows agreement down to $\sim 2T_c$ (see talk by Sylvain Mogliacci)
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Nuclear matter EoSs

Most phenomenologically interesting applications of low-\( T \) strongly interacting matter at small or moderate \( r \) ⇒ Role of weak coupling techniques mainly to constrain low energy EoSs
Nuclear matter EoSs

Most phenomenologically interesting applications of low-$T$ strongly interacting matter at small or moderate $r \Rightarrow$ Role of weak coupling techniques mainly to constrain low energy EoSs

Low energy nuclear physics not derivable from first principles, but experimentally under excellent control $\Rightarrow$ Several model EoSs, which agree well at low densities

However: Many uncertainties with increasing density
  - Composition of the matter: Hyperons, Kaon condensation,...
  - Multi-nucleon interactions (often ignored)
  - Form of variational ansatz
  - Details of hyperon interactions, kaon condensation potential, etc.
Cold quark matter

Introduction: Nuclear matter EoS

Nuclear matter EoSs

\[ \theta \approx \pi/2 \]

The QCD equation of state
Quark matter EoS

At high densities, gauge coupling small ⇒ Use weak coupling techniques to evaluate EoS

Problem: Effects of quark pairing important to take into account ($\Delta \sim e^{-\#/g}$), but formulating weak coupling calculations with anomalous propagators and vertices difficult

- At asymptopia, physical phase Color-Flavor-Locking (CFL)
- Two competing effects: Pairing increases pressure, but deformation of Fermi surfaces decreases it

Leading order solution: Add condensation energy term to the pressure of unpaired quark matter

$$p = p_{\text{pert}} + \# \times \frac{\Delta^2 \mu_B^2}{3\pi^2}$$
Complication in evaluation of $p_{\text{pert}}$: For practical applications, must keep strange quark mass non-zero — state of the art three loops (Kurkela, Romatschke, AV)

Also need to enforce $\beta$-equilibrium and charge neutrality
Quark matter EoS

Complication in evaluation of $p_{\text{pert}}$: For practical applications, **must keep strange quark mass non-zero** — state of the art three loops (Kurkela, Romatschke, AV)
Interpolation to intermediate densities

For densities relevant for neutron star interiors, no quantitatively reliable method available

- Nuclear matter EoSs differ wildly
- Weak coupling expansions show poor convergence

In particular, details of the phase transition remain unknown: $\mu_c$, existence of mixed phase,...

Two possibilities:

1. Model calculations based on symmetries
   - Conceptually appealing, but validity hard to estimate quantitatively

2. Interpolation between trusted limits, requiring thermodynamically stable matching
   - Hope: Bulk thermo insensitive to details of phase structure
Interpolation to intermediate densities

Result of thermodynamic matching: EoS band for all densities; (Kurkela, Romatschke, AV, Wu)
Interpolation to intermediate densities

Ultimately, mass-radius measurements of neutron stars will determine the correct EoS
Table of contents

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(Resummed) perturbation theory important tool for description of equilibrium thermodynamics of deconfined QCD matter because

- Finite density no obstacle: Existing $\mu = 0$ results generalizable (and indeed, generalized) to $\mu \neq 0$
- Provides information on the approach of the system towards free theory limit as $T \rightarrow \infty$
- Only available tool for very high density quark matter — extremely useful constraint of nuclear matter EoSs

Status of perturbation theory well established at $\theta \lesssim 1$, much less so at low temperatures

- Main open challenge: Correct incorporation of quark pairing effects into a high order weak coupling calculation