Construction of multiquark states

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Outline

- Decomposition of Tensor Representations of $SU(m)$
- Yamanouchi Basis
- Application to Pentaquarks
- Remarks

Decomposition of Tensor Representations of $SU(m)$ Groups

- n-quark states $|q_1\rangle|q_2\rangle\cdots|q_n\rangle$ form a m^n dimensional direct product basis of $SU(m)$ ($m = 3,\ 3,\ 2$ for the color, flavor, and spin).
- The direct product representations of $SU(m)$ can be decomposed according to the irreducible representations of the permutation group S_n

$$\boxed{1}\otimes\boxed{2} = \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\oplus\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$$

$$\boxed{1}\otimes\boxed{2}\otimes\boxed{3} = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\oplus\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\oplus\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\oplus\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

$$\boxed{1}\otimes\boxed{2}\otimes\boxed{3}\otimes\boxed{4} = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}\oplus\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}\oplus\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}\oplus\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$$

$$\oplus\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\oplus\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}\oplus\begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}\oplus\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\oplus\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\oplus\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$$

Yamanouchi Basis for Multiquark Systems

Yamanouchi Basis is also called the standard basis in permutation group. For q^2, q^3 and q^4 systems, for example, the basis functions are defined as

$$\psi_S = \boxed{1\,|\,2} = |[2]\,(11)\rangle, \quad \psi_A = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} = |[11]\,(21)\rangle$$

$$\psi_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array} = |[21]\,(211)\rangle, \quad \psi_\rho = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array} = |[21]\,(121)\rangle$$

$$\psi_S = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} = |[3]\,(111)\rangle, \quad \psi_A = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = |[111]\,(321)\rangle$$

$$\psi_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} 4 & \\ \cline{1-1} \end{array} = |[211]\,(3211)\rangle, \quad \psi_\rho = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} 4 & \\ \cline{1-1} \end{array} = |[211]\,(3121)\rangle$$

Yamanouchi Basis in General

- In general, a Yamanouchi basis function is written as

$$|[\lambda_1, \lambda_2, \cdots]\,(r_n, r_{n-1}, \cdots, r_2, r_1)\rangle$$

λ_i: the number of boxes in the ith row of a Young tabloid;
r_i: from the ith row a box is removed.

- Each Young tableau leads to one Yamanouchi basis function, for example,

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} = |[321](322111)\rangle, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} = |[321](321121)\rangle$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} = |[321](312121)\rangle, \quad \cdots$$

- All such defined functions for a Young tabloid together form a complete basis.

Representations of S_n under Yamanouchi basis

It is to evaluate the matrices for all permutations of the permutation group S_n under Yamanouchi basis

- Suppose that the irreducible representations of S_{n-1} are known, then we have the matrices for any element which is in both the S_{n-1} and S_n, for example, the permutation $(i, n-1)$.
- Any element of S_n can be resolved into a product of transpositions (i, j) (for example, $(123) = (13)(12)$), thus what we need to evaluate are the matrices of the elements (i, n).
- But due to

$$(i, n) = (n-1, n)(i, n-1)(n-1, n)$$

we need to evaluate only the matrices for the element $(n-1, n)$.

Representations of S_n under Yamanouchi basis

The operation of the element $(n-1,n)$ on the standard basis satisfies the followings:

$$\mathbf{A}: \qquad (n-1,n)\,|[\lambda](r,r,\cdots)\rangle = +\,|[\lambda](r,r,\cdots)\rangle.$$

$$\mathbf{B}: \qquad (n-1,n)|[\lambda](r,r-1,\cdots)\rangle = -\,|[\lambda](r,r-1,\cdots)\rangle$$

when $|[\lambda](r-1,r,r_{n-2},\cdots,r_2,1)\rangle$ not exist

$$\mathbf{C}: \qquad (n-1,n)|[\lambda](r,s,\cdots)\rangle = \sigma_{rs}|[\lambda](r,s,\cdots)\rangle + \sqrt{1-\sigma_{rs}^2}|[\lambda](s,r,\cdots)\rangle$$

when $r \neq s$. For $[\lambda] = [\lambda_1, \lambda_2, \cdots, \lambda_r \cdots \lambda_s \cdots \lambda_n]$, we have

$$\sigma_{rs} = \frac{1}{(\lambda_r - r) - (\lambda_s - s)}$$

Representations for S_3

- $\psi_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} = |[21]\,(211)\rangle$ and $\psi_\rho = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} = |[21]\,(121)\rangle$ form a complete basis.
- Matrices for all permutations of S_3 in this basis are

$$D(12) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D(23) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$D(13) = D(23)D(12)D(23) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$D(123) = D(13)D(12) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$D(132) = D(12)D(13) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

Projection Operators

Projection operators of S_n are defined in the form

$$W^{[\lambda]}_{(r)} = \sum_i \langle[\lambda](r)|R_i|[\lambda](r)\rangle R_i$$

R_i: all the permutations of S_n

- $W^{[\lambda]}_{(r)}$: projection operator corresponding to the irreducible representation $[\lambda]$ and the Yamanouchi basis function $|[\lambda](r)\rangle$ of S_n.
- For q^3 system, the projection operators according to each Young tableau

$$P^S = 1 + (12) + (13) + (23) + (123) + (132)$$

$$P^\lambda = 1 + (12) - \frac{1}{2}(13) - \frac{1}{2}(23) - \frac{1}{2}(123) - \frac{1}{2}(132)$$

$$P^\rho = 1 - (12) + \frac{1}{2}(13) + \frac{1}{2}(23) - \frac{1}{2}(123) - \frac{1}{2}(132)$$

$$P^A = 1 - (12) - (13) - (23) + (123) + (132)$$

q^3 Flavor Wave Functions

Acting $W^{[\lambda]}_{(r)}$ on any function $f_1 f_2 \cdots f_n$, one could derive the corresponding standard basis function. Let act, for instance, the operator P^λ and P^ρ onto the state udu (with $u \equiv \phi_u$ and $d \equiv \phi_d$), we have

$$\begin{aligned} P^\lambda udu &= udu + duu - \frac{1}{2}udu - \frac{1}{2}uud - \frac{1}{2}duu - \frac{1}{2}uud \\ &= \frac{1}{2}udu + \frac{1}{2}duu - uud \\ \Longrightarrow \quad \psi_\lambda &= \frac{1}{\sqrt{6}}\left[udu + duu - 2\,uud\right] \end{aligned}$$

$$\begin{aligned} P^\rho udu &= udu - duu + \frac{1}{2}udu + \frac{1}{2}uud - \frac{1}{2}duu - \frac{1}{2}uud \\ &= \frac{3}{2}udu - \frac{3}{2}duu \\ \Longrightarrow \quad \psi_\rho &= \frac{1}{\sqrt{2}}\left[udu - duu\right] \end{aligned}$$

q^3 Spin-Flavor States

Spin-flavor wave functions of various permutation symmetries may be written in the general form,

$$\Psi_{S,A,\lambda,\rho} = \sum_{i=S,A,\lambda,\rho} \sum_{j=S,A,\lambda,\rho} a_{ij}\, \psi_i\, \chi_j$$

The coefficient a_{ij} can be determined by applying the permutation operators of S_3 to the general form. Check the simplest case,

$$\begin{aligned} &(23)\,(a\,\psi_\lambda\,\chi_\lambda + b\,\psi_\rho\,\chi_\rho) \\ = a&\left(-\tfrac{1}{2}\psi_\lambda + \tfrac{\sqrt{3}}{2}\psi_\rho\right)\left(-\tfrac{1}{2}\chi_\lambda + \tfrac{\sqrt{3}}{2}\chi_\rho\right) + b\left(\tfrac{1}{2}\psi_\rho + \tfrac{\sqrt{3}}{2}\psi_\lambda\right)\left(\tfrac{1}{2}\chi_\rho + \tfrac{\sqrt{3}}{2}\chi_\lambda\right) \\ &= \left(\tfrac{1}{4}a + \tfrac{3}{4}b\right)\psi_\lambda\chi_\lambda + \left(\tfrac{3}{4}a + \tfrac{1}{4}b\right)\psi_\rho\chi_\rho - \tfrac{\sqrt{3}}{4}(a-b)\,(\psi_\lambda\chi_\rho + \psi_\rho\chi_\lambda) \end{aligned}$$

$a = b$ leads to the fully symmetric spin-flavor wave function. Here we have used $D^{[21]}(23)$, the $[21]$ representation matrix for the element (23) of S_3,

$$D^{[21]}(23) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

$q^4\,\overline{q}$ Systems

- As the color part of the antiquark in pentaquark states is a $[11]$ antitriplet, the color wave function of the four-quark cluster must be a $[211]$ triplet

$$\psi^c_{[211]}$$

- That the total wave function of the four quark configuration is antisymmetric dictates the q^4 orbital-spin-flavour part must be a [31] state,

$$\psi^{osf}_{[31]}$$

- Total wave function of the q^4 configuration may be written in the general form

$$\psi = \sum_{i,j=\lambda,\rho,\eta} a_{ij} \psi^c_{[211]_i} \psi^{osf}_{[31]_j}$$

- Considering the results for q^3 systems, an antisymmetric Ψ may be formed by only three components, that is

$$\psi = a_{\lambda\rho} \phi^c_{[211]_\lambda} \psi^{osf}_{[31]_\rho} + a_{\rho\lambda} \psi^c_{[211]_\rho} \psi^{osf}_{[31]_\lambda} + a_{\eta\eta} \psi^c_{[211]_\eta} \psi^{osf}_{[31]_\eta}.$$

Antisymmetric $q^4\,\overline{q}$ Wave Function

Applying the permutation (34) of S_4, we have

$$
\begin{aligned}
(34)\psi &= -a_{\lambda\rho}\psi^{c}_{[211]_\lambda}\psi^{osf}_{[31]_\rho} \\
&+ a_{\rho\lambda}\left(-\frac{1}{3}\psi^{c}_{[211]_\rho}+\frac{2\sqrt{2}}{3}\psi^{c}_{[211]_\eta}\right)\left(\frac{1}{3}\psi^{osf}_{[31]_\lambda}+\frac{2\sqrt{2}}{3}\psi^{osf}_{[31]_\eta}\right) \\
&+ a_{\eta\eta}\left(\frac{2\sqrt{2}}{3}\psi^{c}_{[211]_\rho}+\frac{1}{3}\psi^{c}_{[211]_\eta}\right)\left(\frac{2\sqrt{2}}{3}\psi^{osf}_{[31]_\lambda}-\frac{1}{3}\psi^{osf}_{[31]_\eta}\right).
\end{aligned}
$$

An antisymmetric Ψ demands

$$a_{\rho\lambda} = -a_{\eta\eta}$$

Here we have used the $[31]$ and $[211]$ representation matrices for the permutation (34) of the S_4,

$$
D^{[31]}(34) = \begin{pmatrix} -1/3 & 2\sqrt{2}/3 & 0 \\ 2\sqrt{2}/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix},\; D^{[211]}(34) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/3 & 2\sqrt{2}/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{pmatrix}
$$

Antisymmetric $q^4\,\overline{q}$ Wave Function

Applying the permutation (12) or (23) of the S_4 leads to

$$a_{\rho\lambda} = -a_{\lambda\rho}$$

Finally, we derive a fully antisymmetric wave function for the q^4 configuration

$$\psi = \frac{1}{\sqrt{3}}\left(\psi^c_{[211]_\lambda}\psi^{osf}_{[31]_\rho} - \psi^c_{[211]_\rho}\psi^{osf}_{[31]_\lambda} + \psi^c_{[211]_\eta}\psi^{osf}_{[31]_\eta}\right)$$

For ground state pentaquarks, the total wave function takes the form

$$\Psi = \frac{1}{\sqrt{3}}\left[\Psi^o_{[5]}\left(\Psi^c_{[211]_\lambda}\Psi^{sf}_{[31]_\rho} - \Psi^c_{[211]_\rho}\Psi^{sf}_{[31]_\lambda} + \Psi^c_{[211]_\eta}\Psi^{sf}_{[31]_\eta}\right)\right]$$

- Ψ^o, Ψ^c and Ψ^{sf} are respectively the orbital, color and spin-flavor parts of the pentaquark states.
- The subscripts $[211]$ and $[31]$ of the Ψ^c and Ψ^{sf} stand for the symmetries of the q^4 configuration in pentaquark states.

q^4 Spin-Flavor Wave Functions

For the pentaquark states with isospin $I = 0$ and strangeness $S = 1$, the flavor-spin wave function of the q^4 configuration must be as follows:

$$\begin{array}{ccccc} [31] & = & [22] & \otimes & [31] \\ SU_{sf}(6) & & SU_f(3) & & SU_s(2) \end{array}$$

Again, the spin-flavor wave functions of various permutation symmetries take the general form,

$$\psi^{\rm sf} = \sum_{i=\lambda,\rho} \sum_{j=\lambda,\rho,\eta} a_{ij}\, \phi_{[22]_i} \chi_{[31]_j}$$

a_{ij} can be determined by acting the permutations of S_4 on the general form. The spin-flavor wave functions for the q^4 cluster are derived as,

$$\begin{aligned} \psi^{\rm sf}_{[31]_\rho} &= -\frac{1}{2}\phi_{[22]_\rho}\chi_{[31]_\lambda} - \frac{1}{2}\phi_{[22]_\lambda}\chi_{[31]_\rho} + \frac{1}{\sqrt{2}}\phi_{[22]_\rho}\chi_{[31]_\eta} \\ \psi^{\rm sf}_{[31]_\lambda} &= -\frac{1}{2}\phi_{[22]_\rho}\chi_{[31]_\rho} + \frac{1}{2}\phi_{[22]_\lambda}\chi_{[31]_\lambda} + \frac{1}{\sqrt{2}}\phi_{[22]_\lambda}\chi_{[31]_\eta} \\ \psi^{\rm sf}_{[31]_\eta} &= \frac{1}{\sqrt{2}}\phi_{[22]_\rho}\chi_{[31]_\rho} + \frac{1}{\sqrt{2}}\phi_{[22]_\lambda}\chi_{[31]_\lambda} \end{aligned}$$

Spin and Flavor Wave Functions

The explicit form of the spin and flavor wave functions of the q^4 configuration of pentaquark states can be easily worked out in the Yamanouchi technique, following the process:

- Work out first the representation matrices in the Yamanouchi basis of the irreducible representations of S_4.
  $D^{[4]}(R_i)$, $D^{[1111]}(R_i)$ (one dimensional matrices)
  $D^{[22]}(R_i)$ (two dimensional matrices)
  $D^{[31]}(R_i)$, $D^{[211]}(R_i)$ (three dimensiuional matrices)

- Construct the corresponding projection operators.
  P_S for $[4]$
  P_A for $[1111]$
  P_λ and P_ρ for $[22]$
  P_λ, P_ρ and P_η for $[31]$ and $[211]$

- Act the projection operators on arbitrary four quark states to obtain the spin and flavor wave functions with the corresponding symmetries.

Flavor Wave Functions

The λ-type projection operator for the representation $[22]$ is derived as

$$\begin{aligned}
P_\lambda &= \sum_{i=1}^{24} \langle [22](2211)|R_i|[22](2211)\rangle R_i \\
&= 2 + 2(12) - (13) - (14) - (23) - (24) + 2(34) \\
&\quad +2(12)(34) + 2(14)(23) + 2(13)(24) \\
&\quad -(123) - (124) - (132) - (134) - (142) - (143) - (234) - (243) \\
&\quad -(1234) - (1243) + 2(1324) - (1342) + 2(1423) - (1432)
\end{aligned}$$

The flavor wave functions of the four-quark subsystem with the [22] symmetry could be derived by operating $P_{\lambda,\rho}$ on any q^4 state. For example,

$$P_\rho(uudd) \Longrightarrow \phi_{[22]_\rho} = \frac{1}{2}(dudu - duud + udud - uddu)$$

$$P_\lambda(uudd) \Longrightarrow \phi_{[22]_\lambda} = \frac{1}{2\sqrt{3}}(duud + udud - 2uudd + uddu + dudu - 2dduu)$$

Spin Wave Functions

The η-type projection operator for the representation $[31]$ is derived as

$$\begin{aligned} P_\eta &= \sum_{i=1}^{24} \langle [31](2111)|R_i|[31](2111)\rangle R_i \\ &= 3 + 3(12) + 3(13) - (14) + 3(23) - (24) - (34) \\ &\quad -(12)(34) - (14)(23) - (13)(24) \\ &\quad +3(123) - (124) + 3(132) - (134) - (142) - (143) - (234) - (243) \\ &\quad -(1234) - (1243) - (1324) - (1342) - (1423) - (1432) \end{aligned}$$

The spin wave functions of the four-quark subsystem with the [31] symmetry can be derived by operating $P_{\lambda,\rho,\eta}$ on any q^4 spin state, for example, the state $\uparrow\uparrow\uparrow\downarrow$,

$$P_\eta(\uparrow\uparrow\uparrow\downarrow) \implies \chi_{[31]_\eta}(1,1) = \frac{1}{2\sqrt{3}} \mid \downarrow\uparrow\uparrow\uparrow + \uparrow\downarrow\uparrow\uparrow + \uparrow\uparrow\downarrow\uparrow - 3 \uparrow\uparrow\uparrow\downarrow \rangle$$

$$P_\rho(\uparrow\uparrow\uparrow\downarrow) \implies \chi_{[31]_\rho}(1,1) = \frac{1}{\sqrt{2}} \mid \downarrow\uparrow\uparrow\uparrow - \uparrow\downarrow\uparrow\uparrow \rangle$$

$$P_\lambda(\uparrow\uparrow\uparrow\downarrow) \implies \chi_{[31]_\lambda}(1,1) = \frac{1}{\sqrt{6}} \mid \downarrow\uparrow\uparrow\uparrow + \uparrow\downarrow\uparrow\uparrow - 2 \uparrow\uparrow\downarrow\uparrow \rangle$$

Yamanouchi basis approach is the very tool for constructing multiquark states.

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